NOTE ON THE SHAPE OF LEVEL CURVES OF GREEN'S FUNCTION

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The object of this note is to indicate some easily proved but geometrically striking properties of the level curves of Green's function. These properties, like convexity and star-shapedness, if possessed by the boundary of a given region, are possessed also by all the level curves; the properties serve to measure the relative nearness to circular shape of the boundary or a level curve. It turns out that in a precise manner the level curves become more nearly circular as they approach the pole of Green's function.

If O is the origin and Γ a smooth arc, the *vector-tangent angle* ψ of Γ at a point P of Γ is defined as the directed angle from the radius vector OP extended beyond P, to the tangent to Γ at P in the positive sense of the curve. The arc Γ is an arc of a circle with center O if and only if we have $\psi \equiv \pi/2$; the nearness of ψ to $\pi/2$ is an indication of the nearness of Γ to circular shape, and if Γ is closed may be measured either by $[\max \psi - \min \psi]$ or by $\max |\psi - \pi/2|$.

The angle $\psi - \pi/2$ has been used recently by P. Davis and Henry Pollak* as a measure of nearness to circularity of Γ *at the point* P, thus considered as a function of both Γ and P. These writers establish part (c) of Theorem 1, which is merely a rephrasing of an inequality due to Grunsky,† and establish a weaker form of Theorem 3, namely a form restricted to bounded simply connected regions. They also point out that the level curves become nearly circular as they approach the pole of Green's function, but do not mention the monotonic character of this variation.

THEOREM 1. *Let the simply connected region R of the $w(=u+iv)$-plane contain $w=0$ but not $w=\infty$, and let $G(u, v)$ be Green's function for R with pole in O. Let $C_r (0<r<1)$ denote the level locus $G(u, v) = -\log r$ in R.* (a) *If the boundary of R is a smooth Jordan curve C_1 for which the vector-tangent angle ψ satisfies the inequalities $\psi_0 \leqq \psi \leqq \psi_1$, the same inequalities hold for the vector-tangent angle of every C_r. Except in the trivial case that R is a circle with center O, we have with $r_1 < r_2 \leqq 1$,* $\max [\psi \text{ on } C_{r_1}] < \max [\psi \text{ on } C_{r_2}]$, $\min [\psi \text{ on } C_{r_1}] > \min [\psi \text{ on } C_{r_2}]$. (b) *If R is star-shaped, then for the level locus C_r we have*

$$\left|\psi - \frac{\pi}{2}\right| \leqq \tan^{-1} \frac{2r}{1-r^2}. \tag{1}$$

(c) *If R is arbitrary, for C_r we have*

$$\left|\psi - \frac{\pi}{2}\right| \leqq \log \frac{1+r}{1-r}. \tag{2}$$

* Trans. Amer. Math. Soc. vol. 72, 1952, pp. 82–103; Theorem 10 and Corollary 10.1.

† H. Grunsky, Jahresber, d. d. Math. Vereinigung, vol. 43, 1933, pp. 140–143; see also G. M. Golusin, Mat. Sbornik, vol. 1 (43), 1936, pp. 127–135.

Of course ψ is properly not single valued, but for each curve such as C_r we choose $0 \leqq \psi \leqq \pi$ at the first intersection of the prime vector with the curve, and determine ψ at other points of the curve by continuity.

If the function $w=f(z)\equiv z+a_2z^2+\cdots$ maps conformally $|z|<1$ onto R, C_r is the image (Kreisbild) of $|z|=r<1$. It is no loss of generality here to assume $f'(0)=1$. The inverse of $w=f(z)$ is $z=F(w)\equiv e^{-G(u,v)-iH(u,v)}$, where $H(u, v)$ is conjugate to $G(u, v)$ in R, so the circle $|z|=r$ corresponds to C_r: $|F(w)|\equiv e^{-G(u,v)}=r$. We have at any point of C_r, in the notation $z=re^{i\theta}$, $dz=izd\theta$, $dw=f'(z)dz$,

$$\psi = \arg dw - \arg w = \arg\left[\frac{izf'(z)}{f(z)}\right] = \frac{\pi}{2} + \operatorname{Im}\left[\log\frac{zf'(z)}{f(z)}\right]; \tag{3}$$

this last member is a function harmonic in $|z|<1$ even at $z=0$ (when suitably defined there), by virtue of $f'(z)\neq 0$, $f(z)/z\neq 0$, and under the hypothesis of (a) is continuous in $|z|\leqq 1$. By the principle of maximum for a harmonic function applied to the region $|z|<1$ we have

$$\max\ [\psi \text{ on } C_r] < \max\ [\psi \text{ on } C_1],$$
$$\min\ [\psi \text{ on } C_r] > \min\ [\psi \text{ on } C_1],$$

except that in the trivial case $zf'(z)/f(z)\equiv 1$ (which implies $f(z)\equiv z$), these inequalities are replaced by equalities. With $r_1<r_2$, the curves C_{r_1} and C_{r_2} are the images of the circles $|z|=r_1/r_2$ and $|z|=1$ under the transformation $w=f(r_2z)$, so (a) is established.

Let now R be star-shaped; if the boundary C_1 is a smooth Jordan curve we have on C_1 the inequality $0\leqq\psi\leqq\pi$, and hence by (a) the inequality $0<\psi<\pi$ holds on each C_r; if C_1 is not a smooth Jordan curve, it is nevertheless well known that each $C_{r'}(r'<1)$ is star-shaped, so each C_r lies in a star-shaped $C_{r'}(0<r<r'<1)$ and again by (a) we have $0<\psi<\pi$ on C_r. From (3) we may therefore write ($|z|<1$)

$$U(z) \equiv \operatorname{Im}\left[\log\frac{zf'(z)}{f(z)}\right], \qquad -\frac{\pi}{2} < U(z) < \frac{\pi}{2}, \qquad U(0)=0. \tag{4}$$

We need to make use of the

LEMMA. *If the function $U(z)$ is harmonic and in modulus less than $\pi/2$ in the circle $|z|<1$, with $U(0)=0$, then in the circle $|z|\leqq r(<1)$ we have*

$$|U(z)| \leqq \tan^{-1}\frac{2r}{1-r^2}.$$

This lemma is proved with the aid of Schwarz's Lemma, by means of the function

$$\zeta \equiv \xi + i\eta \equiv -i \log \frac{1+iz}{1-iz}, \tag{5}$$

which maps* $|z|<1$ onto the infinite strip S: $|\xi|<\pi/2$ so that $z=0$ corresponds to $\zeta=0$. If $V(z)$ is conjugate to $U(z)$ in $|z|<1$ with $V(0)=0$, then the image S' of $|z|<1$ under the transformation $\zeta=U(z)+iV(z)$ lies in S, and the image S'' of S' under the inverse of (5) lies in $|z|<1$. The composite analytic function $z'=F(z)$, which maps $|z|<1$ onto S'' considered in the z'-plane, satisfies the hypothesis of Schwarz's Lemma, namely $|F(z)|<1$ and $F(0)=0$, so we have $|F(z)|\leqq|z|$ in $|z|<1$. Consequently the image S_r' of $|z|\leqq r(<1)$ under the transformation $\zeta=U(z)+iV(z)$ lies in the image S_r of $|z|\leqq r$ under the transformation (5). Since S is convex, so also is S_r; the transformation (5) is real for real z, hence symmetric in the axis of reals; when $z=\pm r$ we have $\zeta=\pm\tan^{-1}[2r/(1-r^2)]$; the boundary of S_r has vertical tangents at the two latter points, so S_r (and therefore S_r') lies between these tangents; this is the conclusion of the lemma, so (1) follows from (4). The sharpness of inequality (1) is shown by the function $w=f(z)\equiv z/(1-z)^2$, which maps $|z|<1$ into a star-shaped region, namely the w-plane slit along the axis of reals from $w=-1/4$ to $w=-\infty$; for this map we have with the values $z=\pm ir$

$$\psi - \frac{\pi}{2} = \arg\left[\frac{zf'(z)}{f(z)}\right] = \pm \tan^{-1}\frac{2r}{1-r^2}.$$

To prove (c) we use Grunsky's inequality (*loc. cit.*) for schlicht functions ($|z|\leqq r<1$),

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leqq \log\frac{1+r}{1-r}, \tag{6}$$

from which we may write (2) at once.

Inequality (6) is sharp in the sense that with either algebraic sign, for every r there exists some $f(z)$ and some z with $|z|=r$ such that (6) becomes an equality, so (2) is sharp in that same sense.

Of course ψ may be large for a smooth curve which spirals a large but finite number of times about a point different from O. Obviously the second member of (2), like that of (1), approaches zero with r.

To study the shape of the level curves of Green's function for an infinite region with pole at infinity, we may apply Theorem 1 with an inversion in O; the new result obtained is the precise analogue of Theorem 1, and can easily be formulated by the reader.

It is of interest to study the shape of level curves without direct reference to the conformal map. Here it is a convenience to study an infinite region, which need not be simply connected:

* See for instance Carathéodory, *Conformal Representation*, Cambridge University Press, 1932, §52.

THEOREM 2. *Let R be an infinite region whose boundary B is finite, and let $G(u, v)$ be Green's function for R with pole at infinity. If the circle C with center O and radius ρ_0 contains B, then on any arc of a level curve exterior to a concentric circle of radius $\rho(>\rho_0)$ we have*

$$\left|\psi - \frac{\pi}{2}\right| \leqq \sin^{-1}\frac{\rho_0}{\rho}.$$

Theorem 2 follows at once from the fact* that the normal to a level curve must cut C. By inversion we have

THEOREM 3. *Let R be a region not necessarily finite with boundary B, and let $G(u, v)$ be Green's function for R with pole in the finite point O. If B lies exterior to a circle C with center O and radius ρ_0, then on any arc of a level curve interior to a concentric circle of radius $\rho(\rho<\rho_0)$ we have*

$$\left|\psi - \frac{\pi}{2}\right| \leqq \sin^{-1}\frac{\rho}{\rho_0}.$$

The results of the present note indicate relative nearness to circular shape of the level curve C_r as measured globally but in terms of infinitesimal properties. A measure referring to global behavior without infinitesimal properties can be defined as

$$\frac{\min\ |w| \text{ on } C_r}{\max\ |w| \text{ on } C_r},$$

a number which is not greater than unity, and which can be shown to increase monotonically as r decreases and to approach unity as r approaches zero.†

THE GENERAL ELECTRIC MATHEMATICS FELLOWSHIP PROGRAM

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A cooperative educational program involving industry and an educational institution is always an important event and deserves widespread publicity. When the program concerns the field of mathematics, it becomes notable as illustrating the great need for mathematics in the industries of today. Consequently, a brief description of the General Electric Mathematics Fellowship Program, as administered by Rensselaer Polytechnic Institute and the Educational Services Division of the General Electric Company, should be of interest to many engaged in mathematical work.

* Walsh, this MONTHLY, vol. 42, 1935, pp. 1–17.

† Walsh, this MONTHLY, vol. 44, 1937, pp. 202–213.

For several years, the General Electric Company has sponsored two programs in science, each for fifty secondary school teachers, at Union College in Schenectady and at Case Institute of Technology in Cleveland. The success of these two programs led the General Electric Company to consider sponsoring a similar program for secondary school teachers of mathematics. The feasibility of attempting a mathematics program was discussed with officials at Rensselaer Polytechnic Institute and, as a result, the program was initiated with the Head of the Department of Mathematics at Rensselaer as Director.

The General Electric Mathematics Fellowship Program provides for six weeks of summer instruction for fifty secondary school teachers of mathematics from the New England States, New York, New Jersey, Pennsylvania, Delaware, Maryland, Virginia, and the District of Columbia. The Fellowships pay the tuition, board, room, supplies, and round trip fare to Troy from the teaching location of each Fellow.

The aims of the program in mathematics are: (1) to stimulate enthusiasm for the study of mathematics on the part of teachers and, through them, on the part of students and officers of public instruction, (2) to familiarize teachers with some of the important applications of mathematics, (3) to furnish teachers with information which may be of assistance in the guidance of their students, (4) to foster growth in the mathematical knowledge and background of the participants, and (5) to aid these teachers in making their secondary school mathematics courses of vital interest to students.

With these aims in view, the following three courses were specially designed for the General Electric Mathematics Fellows.

S11.01—*Applied Mathematical Analysis.* This course met six times a week. It included the following material: statistics (median, mode, standard deviation, probability, binomial and normal distribution, *etc.*), electricity and magnetism (fundamental definitions, electrostatics, electromagnetism, direct and alternating current circuits, machines), mechanics (fundamental definitions, Newton's Laws, applications).

S11.02—*Topics in Elementary Mathematics from an Advanced Viewpoint.* This course met six times a week. It contained a review of derivatives and the definite integral; trigonometric, exponential, and logarithmic functions (series treatment); elements of projective geometry; the number system.

S11.03—*Mathematics at Work.* This course consisted of two 2-hour laboratory periods a week and, in general, two trips a week to industrial plants and laboratories.

Nine laboratory exercises, to illustrate material from courses S11.01 and S11.02, were scheduled. They included such items as statistics, experiments in electricity, tabular and graphical methods for finding derivatives, mechanical methods for areas and integrals, calculating machines, special coordinate papers, rotating vectors, significant figures.

Eight trips to industrial plants and laboratories comprised the second part of this course. Visits were made to various laboratories of the General Electric

Company and to W. & L. E. Gurley, Troy, maker of surveying and mathematical instruments.

Further educational material was presented by a series of talks on insurance, employment opportunities in science and engineering, and the like; by visits to the Rensselaer computing laboratory and the observatory; and by showing educational films and the General Electric House of Magic.

It will be noted that the program had considerable content. This was a natural result of the decision to have the courses carry graduate credit towards the M.S. degree. This decision was based on the following facts: (1) secondary school teachers in many states need to take courses (or do other work) in order to retain their teaching certificates, (2) it was thought that courses of graduate caliber would be of considerable benefit to those who might have majored in education, (3) it was hoped that some of the Fellows might be encouraged to become candidates for a master's degree.

Applications for the fifty Fellowships were secured by sending announcements of the program to principals and heads of mathematics departments of secondary schools situated in the territory assigned to Rensselaer. These announcements described the program and contained a request that the Fellowships be brought to the attention of all teachers of mathematics. As a result of this coverage, nearly 300 completed applications were submitted.

The selection of individuals to whom Fellowships were to be awarded was made by a committee composed entirely of Rensselaer personnel. The geographical distribution of the Fellows for 1952 is shown herewith: Maine 2; New Hampshire, 2; Vermont, 2; Massachusetts, 8; Connecticut, 2; Rhode Island, 1; New York, 13; New Jersey, 5; Pennsylvania, 8; Delaware, 1; Maryland, 1; Virginia, 3; District of Columbia, 2; total, 50. Of these fifty Fellows, twenty-six were men and twenty-four were women.

The Fellows lived in units of the Institute's dormitories and dined together in one of the dining halls. These arrangements fostered discussions, not only of the program but also of methods and problems of secondary school teaching.

A number of social activities were provided for the Fellows by the Institute and by the General Electric Company. The Fellows also organized many special social events of their own.

In retrospect, it appears that the program was very much worth while. The courses were somewhat difficult, the trips were demanding, and the extracurricular activities were strenuous. In spite of this, there is general agreement that much was accomplished. Several Fellows used the summer experience to aid them in their regular teaching positions, several expressed a determination to do more advanced work in mathematics, several found ideas and applications useful in their regular teaching, and several found their standing enhanced by their holding a Fellowship. The results were sufficiently encouraging for the General Electric Company to decide to start a similar mathematics program at Purdue University in the summer of 1953. This program was available to to teachers in eleven states west of the territory served by Rensselaer.

ON THE UNIQUENESS OF THE DISTRIBUTION FUNCTION FOR THE BUFFON NEEDLE PROBLEM

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In the well known Buffon needle problem, a needle of length L is dropped on a board ruled with equidistant parallel lines of spacing D where $D \geqq L$; it is required to determine the probability that the needle will intersect one of the lines.

The needle position is determined by the distance, x, of its middle point from the nearest line, and by the acute angle, ϕ, between this perpendicular distance and the needle. In one solution of this problem [1], the variables x and ϕ are considered independent, and as a hypothesis, the distribution of probability for x and ϕ is assumed to be uniform. Since the needle will intersect one line if

$$x \leqq \frac{L}{2} \cos \phi,$$

the probability, P, that the needle intersects one of the lines is given by

$$P = \frac{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}L \cos \phi} dx d\phi}{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}D} dx d\phi}.$$

It follows, therefore, that

$$P = \frac{2L}{\pi D}.$$

Since experiments [2, 3] made agree well with this result, it is stated that this indirectly confirms the hypothesis of a uniform distribution in x and ϕ.

Using a very elegant method, and making no assumptions as to the distribution function, Barbier [1], also obtains $P = 2L/\pi D$.

It is the purpose of this paper to consider the distribution functions which lead to this probability.

We are therefore interested in solutions of the integral equation

$$\frac{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}L \cos \phi} F(x, \phi) dx d\phi}{\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}D} F(x, \phi) dx d\phi} = \frac{2L}{\pi D}.$$

The solution of this integral equation is not unique, for there are infinitely

many functions of the form $F(x, \phi) = A + G(\phi)$ which satisfy the integral equation, where $G(\phi)$ satisfies

$$\int_0^{\frac{1}{2}\pi} G(\phi)d\phi = 0 \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} G(\phi)\cos\phi d\phi = 0.$$

One simple example is

$$G(\phi) = B[\cos 2\phi + 2\cos 4\phi].$$

It is clear that the distribution function is not unique.

We will now consider the effects on the uniqueness of the distribution function if x and ϕ are assumed to be independent variables. Then, $F(x, \phi) = G(\phi)H(x)$, and

$$\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}L\cos\phi} G(\phi)H(x)dxd\phi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}D} G(\phi)H(x)dxd\phi} = \frac{2L}{\pi D}.$$

Thus

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}D} G(\phi)H(x)dxd\phi = KD.$$

Differentiating with respect to D yields

$$H(\tfrac{1}{2}D)\int_0^{\frac{1}{2}\pi} G(\phi)d\phi = K.$$

Hence $H(x) =$ constant, and the distribution in x is uniform. Then for $G(\phi)$ we must have

$$\frac{\int_0^{\frac{1}{2}\pi} G(\phi)\cos\phi d\phi}{\int_0^{\frac{1}{2}\pi} G(\phi)d\phi} = \frac{2}{\pi}.$$

The solution of $G(\phi)$ is not unique, as was shown previously:

$$[i.e.,\ G(\phi) = B + C[\cos 2\phi + 2\cos 4\phi]].$$

Consequently, we cannot assume a uniform distribution on the basis that it gives the desired probability. However, if we take a slightly more involved version of the Buffon problem it will be shown that there is only one distribution function which leads to the probability one would obtain if a uniform distribution were assumed.

Let us now consider the needle problem with $L \geqq D$, instead of $L \leqq D$. Define

ϕ_0 as $\cos\phi_0 = D/L$. Then, if $\phi \leqq \phi_0$, the needle will surely intersect at least one line. For $\phi \geqq \phi_0$ the problem is as before. Thus, assuming ϕ and x are independent,

$$(\mathrm{I}) \qquad P = \frac{\int_0^{\phi_0}\int_0^{\frac{1}{2}D} G(\phi)H(x)dxd\phi + \int_{\phi_0}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}L\cos\phi} G(\phi)H(x)dxd\phi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}D} G(\phi)H(x)dxd\phi}.$$

P is the probability we would get if we assumed $G(\phi)H(x) = \text{constant}$, whence

$$P = \frac{2\phi_0}{\pi} + \frac{2L}{\pi D}[1 - \sin\phi_0].$$

Now we wish to show that there is only one distribution function which leads to this probability, P.

Replace D by $L\cos\phi_0$, so that we can consider ϕ_0 and L as independent variables. Differentiating (I) partially with respect to L yields

$$\int_{\phi_0}^{\frac{1}{2}\pi} G(\phi)H(\tfrac{1}{2}L\cos\phi)\cos\phi d\phi = H(\tfrac{1}{2}L\cos\phi_0)M(\phi_0).$$

Differentiating this result with respect to ϕ_0 yields

$$-G(\phi_0)H(\tfrac{1}{2}L\cos\phi_0)\cos\phi_0 = -\tfrac{1}{2}L\sin\phi_0 H'(\tfrac{1}{2}L\cos\phi_0)M(\phi_0) + H(\tfrac{1}{2}L\cos\phi_0)M'(\phi_0).$$

Let

$$Y = \tfrac{1}{2}L\cos\phi_0.$$

Then $YH'(Y)/H(Y) = N(\phi_0)$. Since Y and ϕ_0 are independent, it follows that $YH'(Y)/H(Y) = \text{constant}$, and that $H(Y) = AY^{n-1}$ where A and n are arbitrary constants. To determine $(n-1)$ we substitute back in (I) which reduces to

$$\frac{\int_0^{\phi_0} G(\phi)d\phi}{\int_0^{\frac{1}{2}\pi} G(\phi)d\phi} + \frac{\int_{\phi_0}^{\frac{1}{2}\pi} L^n G(\phi)\cos^n\phi d\phi}{\int_0^{\frac{1}{2}\pi} D^n G(\phi)d\phi} = \frac{2\phi_0}{\pi} + \frac{2L}{\pi D}[1 - \sin\phi_0].$$

We replace D by $L\cos\phi_0$, and differentiate with respect to ϕ_0:

$$n\cos^{n-1}\phi_0\sin\phi_0\int_0^{\phi_0} G(\phi)d\phi = \frac{2A}{\pi}[n\phi_0\cos^{n-1}\phi_0\sin\phi_0 + (n-1)\cos^{n-2}\phi_0\sin\phi_0(1-\sin\phi_0)].$$

Here,

$$A = \int_0^{\frac{1}{2}\pi} G(\phi)d\phi.$$

Hence,

$$\frac{\int_0^{\phi_0} G(\phi)d\phi}{\int_0^{\frac{1}{2}\pi} G(\phi)d\phi} = \frac{2\phi_0}{\pi} + \frac{2(n-1)}{\pi}\,\frac{1-\sin\phi_0}{\cos\phi_0}.$$

Letting $\phi_0 = 0$, [*i.e.*, $D = \dot{L}$], we get $n-1=0$. Thus $G(\phi)=$constant, and the distribution is uniform in X and ϕ.

References

1. J. V. Uspensky, Introduction to Mathematical Probability, McGraw-Hill, 1937, p. 251 and p. 254.
2. G. Castelnuovo, Calcolo delle Probabilità, vol. 1, p. 183.
3. A. DeMorgan, A Budget of Paradoxes, vol. 1, p. 282.

SOME PROPERTIES OF FIBONACCI NUMBERS

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In this paper, I consider the series of Fibonacci 1, 2, 3, 5, 8, 13, $\cdots$, the law of formation of whose terms is that any term (from the third onwards) is equal to the sum of the two preceding terms. I denote the terms of this series by

$$U_1, U_2, U_3, U_4, \cdots$$

and I define $U_0 = 1$. I call the U's 'Fibonacci numbers.' Thus any three consecutive Fibonacci numbers are connected by the relation $U_n = U_{n-1} + U_{n-2}$.

Among the several known results concerning Fibonacci numbers, I quote below some interesting ones:

I $$U_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

II $$U_{2n} = U_n^2 + U_{n-1}^2$$

III $$U_{3n+2} = U_{n+1}^3 + U_n^3 - U_{n-1}^3$$

IV $\quad U_n^2 - U_{n-p}U_{n+p} = (-1)^{n-p+1}U_{p-1}^2$

V $\quad U_1 + U_2 + U_3 + \cdots + U_n = U_{n+2} - 2$

VI $\quad U_1 - U_2 + U_3 - \cdots + (-1)^{n-1}U_n = (-1)^{n-1}U_{n-1}.$

I now list below some results which can all be proved by the principle of mathematical induction:

$$U_1 + U_3 + U_5 + \cdots + U_{2n-1} = U_{2n} - 1$$
$$U_2 + U_4 + U_6 + \cdots + U_{2n} = U_{2n+1} - 1$$
$$U_1 + U_4 + U_7 + \cdots + U_{3n-2} = \tfrac{1}{2}(U_{3n} - 1)$$
$$U_2 + U_5 + U_8 + \cdots + U_{3n-1} = \tfrac{1}{2}(U_{3n+1} - 1)$$
$$U_3 + U_6 + U_9 + \cdots + U_{3n} = \tfrac{1}{2}U_{3n+2} - 1$$
$$U_{2n+1} = U_{n+1}^2 - U_{n-1}^2$$
$$U_1 + U_5 + U_9 + \cdots + U_{4n-3} = U_{2n-1}^2$$
$$U_3 + U_7 + U_{11} + \cdots + U_{4n-1} = U_{2n}^2 - 1$$
$$U_2 + U_6 + U_{10} + \cdots + U_{4n-2} = \tfrac{1}{2}(U_{4n-1} + U_{2n-1}^2)$$
$$U_4 + U_8 + U_{12} + \cdots + U_{4n} = \tfrac{1}{2}(U_{4n+3} - U_{2n+1}^2 - 2)$$
$$U_nU_{n+2} = U_{n+1}^2 + (-1)^n$$
$$U_1^2 + U_2^2 + U_3^2 + \cdots + U_n^2 = \tfrac{1}{2}(U_{2n+3} - U_{n+1}^2 - 2)$$
$$= U_nU_{n+1} - 1$$
$$U_2^2 + U_4^2 + U_6^2 + \cdots + U_{2n}^2 = \tfrac{1}{5}(3U_{2n+1}^2 + 2U_{2n}^2 - 4U_{2n-1}U_{2n+1} + 2n - 5)$$
$$U_1^2 + U_3^2 + U_5^2 + \cdots + U_{2n-1}^2 = \tfrac{1}{5}(3U_{2n}^2 + 2U_{2n+1}^2 - 6U_{2n-1}U_{2n+1} - 2n - 5)$$
$$U_1U_3 + U_3U_5 + \cdots + U_{2n-1}U_{2n+1} = \tfrac{1}{5}(3U_{2n+1}^2 - 2U_{2n}^2 - 3n - 1)$$
$$U_0U_2 + U_2U_4 + \cdots + U_{2n-2}U_{2n} = \tfrac{1}{5}(2U_{2n+1}^2 - 3U_{2n}^2 + 3n + 1)$$
$$U_0U_2 + U_1U_3 + \cdots + U_{2n-2}U_{2n} + U_{2n-1}U_{2n+1} = U_{2n}U_{2n+1} - 1$$
$$U_1U_2 + U_3U_4 + \cdots + U_{2n-1}U_{2n} = \tfrac{1}{5}(U_{2n+1}^2 + U_{2n}^2 - n - 2)$$
$$U_0U_1 + U_2U_3 + \cdots + U_{2n-2}U_{2n-1} = \tfrac{1}{5}(4U_{2n}^2 - U_{2n+1}^2 + n - 3)$$
$$U_0U_1 + U_1U_2 + \cdots + U_{2n-2}U_{2n-1} + U_{2n-1}U_{2n} = U_{2n}^2 - 1.$$

Finally the congruence relation

$$U_{(n+1)k+n} \equiv 0 \pmod{U_n}$$

may be proved by induction on k.

By using the identity

$$U_m^2 - U_{m-p}U_{m+p} = (-1)^{m-p+1}U_{p-1}^2,$$

I deduce the following results:

$$U_0U_4 + U_2U_6 + \cdots + U_{2n-2}U_{2n+2} = \tfrac{1}{5}(3U_{2n+1}^2 - 2U_{2n}^2 + 7n - 1)$$
$$U_1U_5 + U_3U_7 + \cdots + U_{2n-1}U_{2n+3} = \tfrac{1}{5}(2U_{2n+3}^2 - 3U_{2n+2}^2 - 7n - 6)$$
$$U_0U_4 + U_1U_5 + \cdots + U_{2n-2}U_{2n+2} + U_{2n-1}U_{2n+3} = \tfrac{1}{2}(U_{4n+5} - U_{2n+2}^2 - 4).$$

By starting with the identities

$$U_mU_{m+1}U_{m+2} = U_{m+1}^3 + (-1)^mU_{m+1}$$

and

$$U_{m+1}^3 = U_m^3 + U_{m-1}^3 + 3U_{m-1}U_mU_{m+1},$$

we can easily prove the following:

$$U_1U_2U_3 + U_3U_4U_5 + \cdots + U_{2n-1}U_{2n}U_{2n+1} = \tfrac{1}{4}(U_{2n+1}^3 - U_{2n+1})$$
$$U_0U_1U_2 + U_2U_3U_4 + \cdots + U_{2n-2}U_{2n-1}U_{2n} = \tfrac{1}{4}(U_{2n}^3 + U_{2n} - 2)$$
$$U_0^3 + U_2^3 + U_4^3 + \cdots + U_{2n}^3 = \tfrac{1}{4}(U_{2n+1}^3 + 3U_{2n+1})$$
$$U_1^3 + U_3^3 + U_5^3 + \cdots + U_{2n-1}^3 = \tfrac{1}{4}(U_{2n}^3 - 3U_{2n} + 2)$$
$$U_1^3 + U_2^3 + U_3^3 + \cdots + U_n^3 = \tfrac{1}{4}(U_{n+1}^3 + U_n^3 + (-1)^n3U_{n-1} - 2).$$

By making use of the identity

$$U_{m+1}^3 = U_{m-1}U_{m+1}U_{m+3} + (-1)^mU_{m+1}$$

and the preceding results, it may be proved that

$$U_0U_2U_4 + U_2U_4U_6 + \cdots + U_{2n-2}U_{2n}U_{2n+2} = \tfrac{1}{4}(U_{2n+1}^3 + 7U_{2n+1} - 8)$$
$$U_1U_3U_5 + U_3U_5U_7 + \cdots + U_{2n-1}U_{2n+1}U_{2n+3} = \tfrac{1}{4}(U_{2n+2}^3 - 7U_{2n+2} + 6)$$
$$U_0U_2U_4 + U_1U_3U_5 + U_2U_4U_6 + \cdots + U_{n-1}U_{n+1}U_{n+3}$$
$$= \frac{1}{10}[U_{3n+7} + (-1)^{n+1}16U_n - 5].$$

I now prove the following main

THEOREM. *If k is an integer >1, then* i) *a positive integer l, less than k and depending on k, can be found such that*

$$U_{n_1+n_2+\cdots+n_k-l} < U_{n_1}U_{n_2}\cdots U_{n_k} < U_{n_1+n_2+\cdots+n_k-l+1}$$

whenever each of the $n_1, n_2, \cdots, n_k$ is $>n_0$, n_0 depending on k. ii) *there lie exactly mk Fibonacci numbers between U_n^k and U_{n+m}^k, n being $>n_0$.*

Proof of (i). Let l be the greatest integer in

$$\left(\frac{k-1}{2}\right)\cdot\frac{\log 5}{\log\left(\frac{1+\sqrt{5}}{2}\right)}-k+2,$$

so that

$$\left(\frac{k-1}{2}\right)\cdot\frac{\log 5}{\log\left(\frac{1+\sqrt{5}}{2}\right)}-k+1<l<\left(\frac{k-1}{2}\right)\cdot\frac{\log 5}{\log\left(\frac{1+\sqrt{5}}{2}\right)}-k+2.$$

We have

$$\begin{aligned}U_{n_1}U_{n_2}\cdots U_{n_k}&=\left(\frac{1}{\sqrt{5}}\right)^k\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n_1+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n_1+1}\right\}\\&\quad\cdots\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n_k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n_k+1}\right\}\\&=\frac{1}{5^{k/2}}\left(\frac{1+\sqrt{5}}{2}\right)^{n_1+n_2+\cdots+n_k+k}\left\{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n_1+1}\right\}\\&\quad\cdots\left\{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n_k+1}\right\}\end{aligned}$$

and

$$\begin{aligned}&U_{n_1+n_2+\cdots+n_k-l+1}\\&\quad=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n_1+n_2+\cdots+n_k-l+2}\left\{1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n_1+n_2+\cdots+n_k-l+2}\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n_1,\cdots,n_k\to\infty}\frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l+1}}&=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+l-2}}{5^{\frac{1}{2}(k-1)}}\\&<\frac{1}{5^{\frac{1}{2}(k-1)}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}(k-1)\log 5/[\log((1+\sqrt{5})/2)]}\\&=\frac{5^{\frac{1}{2}(k-1)}}{5^{\frac{1}{2}(k-1)}}=1.\end{aligned}$$

Therefore

$$\frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l+1}} < 1$$

if each of $n_1, n_2, \cdots, n_k$ is $>m_0$.
Similarly

$$\lim_{n_1,\cdots,n_k\to\infty} \frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+l-1}}{5^{\frac{1}{2}(k-1)}}$$

$$> \frac{1}{5^{\frac{1}{2}(k-1)}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}(k-1)\log 5/[\log((1+\sqrt{5})/2)]} = 1.$$

Therefore

$$\frac{U_{n_1}U_{n_2}\cdots U_{n_k}}{U_{n_1+n_2+\cdots+n_k-l}} > 1$$

if each of $n_1, n_2, \cdots, n_k$ is $>N_0$. Hence, if each of $n_1, n_2, \cdots, n_k$ is $>n_0=\max(m_0, N_0)$ we have (i).

COROLLARY. *Putting $n_1=n_2=\cdots=n_k=n$ in the above inequalities, we have for $n>n_0$,*

$$U_{kn-l} < U_n^k < U_{kn-l+1},$$

where l has the value specified in (i) *above.*

Proof of (ii). From the above corollary, we have

$$U_{kn-l} < U_n^k < U_{kn-l+1},$$

and replacing n by $(n+1)$, we have

$$U_{kn+k-l} < U_{n+1}^k < U_{kn+k-l+1}.$$

Hence between U_n^k and U_{n+1}^k lie the k Fibonacci numbers

$$U_{kn-l+1},\ U_{kn-l+2},\ \cdots,\ U_{kn-l+k}.$$

By extending the argument, it can be deduced that between U_n^k and U_{n+m}^k there lie exactly mk Fibonacci numbers.

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TRACE FUNCTIONS ON ALGEBRAS WITH PRIME CHARACTERISTIC

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The natural trace function on the matrix algebra F_d of degree d over a field F is defined as

$$f((\alpha_{ik})) = \operatorname{tr}((\alpha_{ik})) = \sum_{i=1}^{d} \alpha_{ii}.$$

It satisfies the rules

$$f(a + b) = f(a) + f(b) \tag{1}$$

$$f(\lambda a) = \lambda f(a), \tag{2}$$

$$f(ab) = f(ba), \tag{3}$$

for a, b in F_d, λ in F.

Introducing the Lie-multiplication in F_d by

$$a \circ b = ab - ba, \tag{4}$$

the rules (1)–(3) become equivalent to (1), (2) and

$$f(a \circ b) = 0 \tag{3a}$$

for a, $b \in F_d$.

In an associative ring A of prime characteristic p the identity

$$(a_1 + a_2)^p = a_1^p + a_2^p + \Lambda(a_1, a_2) \tag{5}$$

holds, with $\Lambda(a_1, a_2)$ a certain sum of Lie-products of the form

$$a_{i_1} \circ (a_{i_2} \circ (\cdots (a_{i_{p-1}} \circ a_{i_p}) \cdots)).*$$

If A is commutative then (5) assumes the simpler form

$$(a_1 + a_2)^p = a_1^p + a_2^p. \tag{5a}$$

Let f be a function defined on an associative algebra A over a field F of prime characteristic p and assume that the values of f are in F satisfying (1)–(3). It follows from (1), (3a) and (5) that

$$f((a + b)^p) = f(a^p) + f(b^p) \tag{6}$$

and more generally

$$f\left(\left(\sum_{i=1}^{r} \lambda_i a_i\right)^p\right) = \sum_{i=1}^{r} \lambda_i^p f(a_i^p). \tag{7}$$

* Hans Zassenhaus, Über Lie'sche Ringe mit Primzahl Charakteristik, Hamburger Abhandlungen, Bd. 13, p. 89, 1939.

E.g. for $A = F_d$, $f = \mathrm{tr}$ the matrices $e_{rs} = (\delta_{ir}\delta_{ks})$ form a basis of F_d over F with the multiplication rule $e_{rs}e_{uv} = \delta_{su}e_{rv}$ such that for $a = (\alpha_{ik})$,

$$a = \sum \alpha_{ik}e_{ik}^{*}$$

and, according to (7), (5a),

$$\mathrm{tr}\,(a^p) = \sum \alpha_{ik}^p (\mathrm{tr}\, e_{ik})^p = \sum \alpha_{ii}^p = (\sum \alpha_{ii})^p = (\mathrm{tr}\, a)^p,$$

a rule which may be obtained also by using ordinary tools of matrix theory.

DEFINITION. *A function f defined on an associative algebra A over a field F of prime characteristic p with values in an algebraically closed extension Ω of F is called a trace function A over F if it satisfies the rules* (1)–(3) *and the rule*

$$f(a^p) = f(a)^p \qquad \text{for } a \text{ contained in } A. \tag{8}$$

An example is given by the natural trace function on a matrix algebra of characteristic p. More generally, every representation Δ of an associative algebra A over a field F of characteristic $p>0$ by matrices of degree d with coefficients in Ω leads to the trace function $\mathrm{tr}\,\Delta(a)$ on A over F. This follows from the conditions

$$\begin{aligned} \Delta(a+b) &= \Delta a + \Delta b, \\ \Delta(\lambda a) &= \lambda \Delta a, \\ \Delta(ab) &= \Delta a \cdot \Delta b, \end{aligned} \tag{9}$$

for $a, b \in A$, $\lambda \in F$ which every representation must satisfy. It is the purpose of this paper to show that the only trace functions on A over F are the traces defined by representations, and to give an application to the theory of representations of finite groups for characteristic $p>0$.

From now on A will always be an associative algebra over a field F of characteristic $p>0$.

PROPOSITION 1. *The trace functions on A form a module $T(A/F)$ of characteristic p.*

Proof obvious.

Iterating (8) we obtain the rule

$$f(a^{p^j}) = f(a)^p \qquad (j = 0, 1, 2, \cdots) \tag{8a}$$

for trace functions. In particular,

(8b) if an equation $a^{p^j} = 0$ holds then $f(a) = 0$.

This leads to

* The mere summation symbol indicates summation over all indices occurring twice.

PROPOSITION 2. *Each trace function on A over F vanishes on the radical $R(A)$ of A.*

For any element a of A there must be a non-trivial linear relation between the infinitely many elements $a, a^p, a^{p^2}, \cdots$ say

$$\sum_{j=0}^{m} \lambda_j a^{p^j} = 0, \qquad \text{with } \lambda_j \in F, \lambda_m = 0.$$

For a trace function on A over F it follows that

$$0 = f\left(\sum_{j=0}^{m} \lambda_j a^{p^j}\right) = \sum_{j=0}^{m} \lambda_j f(a)^{p^j}.$$

Hence

PROPOSITION 3. *The values of a trace function on A over F are algebraic over F.*

If A is the ring sum of the algebras A_1 and A_2 over F then every trace function f on A induces a trace function f_i on $A_i (i=1, 2)$. The trace function f on A over F is uniquely determined by the pair f_1, f_2 according to the rule

$$f(a_1 + a_2) = f_1(a_1) + f_2(a_2) \qquad \text{for } a_1 \in A_1, a_2 \in A_2. \tag{9}$$

Conversely any pair of trace functions f_i on A_i $(i=1, 2)$ determines a trace function f on A according to (9). Hence

PROPOSITION 4. *The module formed by the trace functions on a ring sum of two algebras over F is direct sum of the modules formed by the trace functions on the summands.*

For the matrix algebra of degree d over F we obtain for each trace function f the equations

$$f(e_{rs}) = f(e_{rs} \circ e_{ss}) = 0, \qquad f(e_{rr} - e_{ss}) = f(e_{rs} \circ e_{sr}) = 0$$

in case of $r \neq s$. Hence for $a = \sum \alpha_{ik} e_{ik}$ it follows that

$$\begin{aligned} f(a) &= \sum \alpha_{ik} f(e_{ik}) = \sum \alpha_{ii} f(e_{ii}) = \sum \alpha_{ii}(f(e_{ii} - e_{11}) + f(e_{11})) \\ &= \sum \alpha_{ii} f(e_{11}) = \operatorname{tr} a \cdot f(e_{11}). \end{aligned}$$

Furthermore

$$f(e_{11})^p = f(e_{11}^p) = f(e_{11});$$

hence $f(e_{11})$ belongs to the prime field. We have obtained

PROPOSITION 5. *The module of the trace functions on a matrix algebra A of characteristic d is generated by the natural trace function so that there are exactly p trace functions on a matrix algebra.*

Let $b_1, b_2, \cdots, b_n$ be a basis of the associative algebra A over F. There are linear relations

$$b_i \circ b_k = b_i b_k - b_k b_i = \sum \gamma_{ikl} b_l, \tag{10}$$

$$b_i^p = \sum \gamma_{il} b_l \tag{11}$$

with $\gamma_{ikl}, \gamma_{il}$ contained in F. In view of the rules (1)–(3), (8) each trace function f on A over F satisfies the rules

$$f(\sum \lambda_l b_l) = \sum \lambda_l f(b_l), \tag{12}$$

$$\sum \gamma_{ikl} f(b_l) = 0, \tag{13}$$

$$f(b_i)^p = \sum \gamma_{il} f(b_l). \tag{14}$$

Conversely we have

PROPOSITION 6. *If a set of n constants $f(b_1), f(b_2), \cdots, f(b_n)$ belonging to the algebraically closed extension Ω of F satisfies* (13), (14), *then it defines a trace function on A over F according to* (12).

Proof: The rules (1)–(3) follow from (12), (13) by obvious computations. Using (1)–(3), (14) and (7) we obtain

$$\begin{aligned} f((\sum \lambda_i b_i)^{p^\rho}) &= \sum \lambda_i^{p^\rho} f(b_i^{p^\rho}) = \sum \lambda_i^{p^\rho} f(\sum \gamma_{il} b_l) \\ &= \sum \lambda_i^{p^\rho} \sum \gamma_{il} f(b_l) = \sum \lambda_i^{p^\rho} f(b_i)^{p^\rho} \\ &= (\sum \lambda_i f(b_i))^{p^\rho} = (f(\sum \lambda_i b_i))^{p^\rho} \end{aligned}$$

and hence (8).

From propositions 1–6 we have the following

THEOREM ON TRACE FUNCTIONS. *The trace functions on an associative algebra A over a field F of prime characteristic p with values in the algebraically closed extension Ω of F coincide with the trace functions belonging to the representations of A over F by matrices of finite degree with coefficients in Ω. The number of the trace functions on A over F is equal to p^ρ where ρ denotes the number of classes of equivalent absolutely irreducible representations of A over F.*

Proof: We recapitulate that a representation Δ of A over F by matrices of degree d with coefficients in Ω is called absolutely irreducible if there are d^2 matrices linearly independent over Ω among the matrices $\Delta(a)$ with a contained in A. Denoting by A_Ω the result of extending the associative algebra A with basis $b_1, b_2, \cdots, b_n$ over F to an associative algebra with basis $b_1, b_2, \cdots, b_n$ over Ω with the same rules of multiplication, we conclude from proposition 6 that for every trace function f on A over F by the formula

$$f_\Omega(\sum \Lambda_l b_l) = \sum \Lambda_l f(b_l) \qquad \text{with } \Lambda_l \in \Omega, \tag{15}$$

there is defined a trace function f_Ω on A_Ω over Ω. Conversely every trace function on A_Ω over the algebraically closed field Ω has its values, according to proposition 2, entirely in Ω. Hence it induces a trace function on A over F. And this correspondence between the trace functions on A_Ω over Ω and the trace functions on A over F provides an isomorphism between the modules $T(A_\Omega/\Omega)$ and $T(A/F)$. According to McLagan-Wedderburn the difference algebra of A_Ω over its radical decomposes into the ring sum of ρ matrix algebras over Ω where by known results ρ coincides with the number of classes of equivalent absolutely irreducible representations of A over F. Due to proposition 2 the number of trace functions of A_Ω over Ω coincides with the corresponding number for $A_\Omega - R(A_\Omega)$ over Ω. From propositions 4, 5 it follows that this number is p^ρ.

Since each trace function on a matrix algebra over F is obtained by multiplying the natural trace function with an element of the prime field, we conclude that it may be interpreted as the trace of a multiple of the natural representation by matrices. In view of the construction leading to proposition 4 each trace function on a ring sum of finitely many matrix algebras over F is the trace of a certain fully reducible representation. Hence any trace function on A_Ω over Ω is the trace of a sum of irreducible representations of A_Ω over Ω by matrices of finite degree with coefficients in Ω, or what amounts to the same, any trace function on A over F is the trace of the sum of absolutely irreducible representations of A over F, *q.e.d.*

By application of proposition 6 and the main theorem to the group algebra of a finite group over the prime field of characteristic $p>0$ we obtain

PROPOSITION 7. *Let ρ_p denote the number of classes of equivalent absolutely irreducible representations of a finite group G for characteristic $p>0$. There are p^{ρ_p} trace functions on G. They are characterized as functions f on G with values in the algebraic algebraically closed field Ω of characteristic p satisfying*

$$f(xy) = f(yx) \tag{16}$$

$$f(x^p) = f(x^p) \tag{17}$$

for $x, y \in G$.

Equivalent to (16), (17) are the rules

$$f(txt^{-1}) = f(x) \tag{16a}$$

$$f(x^{p^j}) = f(x)^{p^j} \tag{17a}$$

for $x, y \in G$, j a non-negative integer.

DEFINITION. *Two elements a, b of G are p-conjugate if there is an equation*

$$xa^{p^j}x^{-1} = yb^{p^k}y^{-1}$$

with $x, y \in G$ and j, k non-negative integers. We write

$$a \underset{p}{\sim} b$$

to indicate the p-conjugacy of a and b, whereas

$$a \sim b$$

indicates that a and b are conjugate under G.

By definition

$$b \underset{p}{\sim} a \qquad \text{if } a \underset{p}{\sim} b,$$

$$a \underset{p}{\sim} b \qquad \text{if } a \sim b,$$

$$a^{p^j} \underset{p}{\sim} a \qquad (j = 0, 1, 2, \cdots).$$

The relation $a \sim b$ has the 3 properties of an equivalence relation. Due to the identity

$$(xax^{-1})^m = xa^m x^{-1},$$

we have

$$a^m \sim b^m \qquad \text{if } a \sim b$$

and

$$a^m \underset{p}{\sim} b^m \qquad \text{if } a \underset{p}{\sim} b.$$

Furthermore $a \underset{p}{\sim} b$ if and only if $a^{p^j} \sim b^{p^k}$ holds for some non-negative integers j, k. Suppose that we also have $b^{p^l} \sim c^{p^m}$. Then

$$a^{p^{j+l}} = (a^{p^j})^{p^l} \sim (b^{p^k})^{p^l} = (b^{p^l})^{p^k} \sim (c^{p^m})^{p^k} = c^{p^{m+k}},$$

$$a \underset{p}{\sim} c.$$

Hence p-conjugacy satisfies the 3 requirements for an equivalence relation. The elements of G are distributed among classes of p-conjugate elements each of which consists of some classes of conjugate elements. Let the order of G be $n = p^r \cdot n'$ with n' prime to p. Then for $a \in G$ we have $a \underset{p}{\sim} a^{p^r}$ so that the order of a^{p^r} divides n'. Hence each element of G is p-conjugate to an element with order prime to p, *i.e.*, to a p-regular element. Let a be a p-regular element, then from $a^n = 1$ follows $a^{n'} = 1$.

Among the classes of conjugate elements represented by $a, a^p, a^{p^2}, \cdots$ there must be repetitions. Let $a^{p^{d(a)}}$ be the first element conjugate to a previous element say

$$a^{p^{d(a)}} \sim a^{p^i} \qquad \text{with } 0 \leqq i < d(a)$$

and denote by p' a solution of the congruence $pp' \equiv 1(n')$; then in case $i>0$ we have

$$a^{p^{d(a)-1}} = (a^{pp'})^{p^{d(a)-1}} = (a^{p^{d(a)}})^{p'} \sim (a^{p^i})^{p'} = a^{p^{i-1}}$$

contrary to the minimum property of d. Hence $i=0$, $a^{p^{d(a)}} \geqq a$, $a^{p^{2d(a)}} \geqq a^{p^{d(a)}}$, $\cdots$

$$a^{p^{jd(a)}} \sim a \qquad (j = 0, 1, 2, \cdots)$$

$$a^{p^i} \sim a^{p^{i+jd(a)}} \qquad (i, j = 0, 1, 2, \cdots).$$

It follows that there are precisely $d(a)$ classes of conjugate elements containing an element a^{p^i} and these $d(a)$ classes are represented by the elements $a, a^p, \cdots, a^{p^{d(a)-1}}$. Every p-regular element which is p-conjugate to a must be in one of the $d(a)$ classes of conjugate elements represented by $a, a^p, \cdots, a^{p^{d(a)-1}}$.

Now we answer the following question: Let a be p-regular, b an element of order $p^\mu q$ where p does not divide q, and let

$$a^{p^j} \sim b^{p^k} \tag{18}$$

for some non-negative integers j, k. Under which conditions is

$$a^{p^l} \sim b^{p^m}, \qquad \text{with some non-negative integers } l, m? \tag{19}$$

Answer: The necessary and sufficient conditions are

$$m \geqq \mu \text{ and the congruence } l + k \equiv j + m(d(a)).$$

Proof: Assume (19). Since a is p-regular, the same is true for a^{p^l} and b^{p^m}, hence $m \geqq \mu$. Furthermore

$$a^{p^{l+k}} \sim b^{p^{m+k}} \sim a^{p^{j+m}}.$$

But due to the previous considerations two powers a^{p^r} and a^{p^s} are conjugate under G if and only if $r \equiv s(d(a))$. Hence $l+k \equiv j+m(d(a))$. Conversely let $m \geqq \mu$ and $l+k \equiv j+m(d(a))$; then

$$a^{p^{l+k}} \sim a^{p^{j+m}}$$

and b^{p^m} is p-regular,

$$b^{p^{m+k}} = (b^{p^k})^{p^m} \sim (a^{p^j})^{p^m} = a^{p^{j+m}} \sim a^{p^{j+k}},$$

$$b^{p^m} = (b^{p^m})^{(pp')^k} = (b^{p^{m+k}})^{p'^k} \sim (a^{p^{j+k}})^{p'^k} = (a^{(pp')^k})^{p^k} = a^{p^k},$$

which completes the proof.

For the construction of the trace functions on G we choose a representative system $a_1, a_2, \cdots, a_r$ of the classes of p-conjugate elements such that each representative is p-regular. Due to (16a) and (17a) and the relation

$$a_i^{p^{d(a_i)}} \sim a_i,$$

we obtain for each trace function f on G the equations

$$f(a_i)^{p^{d(a_i)}} = f(a_i)$$

which are equivalent to the statement that $f(a_i)$ belongs to the Galois field $GF(p^{d(a_i)})$ of $p^{d(a_i)}$ elements. Furthermore for x contained in G we have the relation

$$x^{p^j} \sim a_i^{p^k} \qquad \text{with some } i, j, k, \tag{20}$$

from which it follows that

$$f(x)^{p^j} = f(a_i)^{p^k} \tag{21}$$

which determines $f(x)$ uniquely once the values $f(a_1), f(a_2), \cdots, f(a_r)$ are known.

Conversely let us assign to each representative a_i one of the $p^{d(a_i)}$ elements of $GF(p^{d(a_i)})$ as the value $f(a_i)$ and let us define $f(x)$ according to (20) and (21) for every element x of G. Due to the answer which we gave to the question above it follows that the value of $f(x)$ is independent of the relation (20) connecting x with the representative a_i. Since $(x^{p^j})^{p^j} \sim a^{p^{k+1}}$ we find that $f(x^{p^j})^{p^j} = f(a_i)^{p^{k+1}}$ and hence $f(x^{p^j}) = f(x)^{p^j}$. If $y \geqq x$ then

$$y^{p^j} \sim x^{p^j} \sim a_i^{p^k},$$
$$f(y)^{p^j} = f(a_i)^{p^k},$$
$$f(y) = f(x).$$

Hence f is a trace function.

According to the previous construction, the number of trace functions on G is p^σ where

$$\sigma = \sum_{i=1}^{r} d(a_i).$$

The exponent of p coincides with the number of classes of conjugate p-regular elements.

Hence we have obtained another proof of the theorem of Richard Brauer* that the number ρ_p of classes of absolutely irreducible representations of a finite group G for characteristic $p>0$ coincides with the number of classes of conjugate p-regular elements.† In addition we have found an explicit construction of the p^{ρ_p} trace functions on G for characteristic p.

* Über die Darstellung von Gruppen in Galois'schen Feldern. Actualités scientifiques et industrielles 195, 1935.

† This application has been suggested to me by a lecture of Professor R. Brauer, in which he makes use of (1), (2), (3) in proving his theorem.

MATHEMATICAL NOTES

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ON INEQUALITIES FOR ANALYTIC FUNCTIONS

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Let $f(z)$ be regular for $|z|<R$. Let

$$A(r) = \max \Re f(z) \quad \text{for} \quad |z| \leqq r, \quad 0 \leqq r < R,$$
$$M(r) = \max |f(z)| \quad \text{for} \quad |z| \leqq r, \quad 0 \leqq r < R,$$
$$A(R) = \lim_{r \to R-0} A(r), \qquad M(R) = \lim_{r \to R-0} M(r).$$

Then the following two inequalities, of which the second includes Cauchy's, may be proved by the use of Schwarz's lemma, as explained by Jensen in a comprehensive paper [1] and by the author in an independent note [2].

$$\frac{|f^{(n)}(0)|}{n!} \leqq \frac{2}{R^n}[A(R) - \Re f(0)], \qquad n \geqq 1. \tag{1}$$

$$\frac{|f^{(n)}(0)|}{n!} \leqq \frac{1}{R^n} \frac{[M(R)]^2 - |f(0)|^2}{M(R)}, \qquad n \geqq 1. \tag{2}$$

The case $n=1$ of each of the inequalities (1), (2) has been generalized as below by Lindelöf [1, pp. 13, 24].

$$|f'(z)| \leqq \frac{2R}{R^2 - r^2}[A(R) - \Re f(z)], \qquad 0 \leqq |z| = r < R. \tag{1'}$$

$$|f'(z)| \leqq \frac{R}{R^2 - r^2} \frac{[M(R)]^2 - |f(z)|^2}{M(R)}, \qquad 0 \leqq |z| = r < R. \tag{2'}$$

An advanced problem recently proposed by H. S. Shapiro [3] suggests the following results deducible from (1), (1'), and supplementary to (2), (2').

THEOREM. *If $f(z)$ is regular and non-vanishing for $|z|<R$ and if $k \geqq 1$ is the least positive integer for which $f^{(k)}(0) \neq 0$, then*

$$\frac{|f^{(n)}(0)|}{n!} \leqq \frac{2}{R^n} \frac{M(R)}{e}, \qquad k \leqq n \leqq 2k-1; \tag{3}$$

and further

$$|f'(z)| \leqq \frac{2R}{R^2 - r^2} \frac{M(R)}{e}, \qquad 0 \leqq |z| = r < R. \tag{3'}$$

Proofs of (3), (3′). To prove (3), consider

$$F(z) = \log f(z)$$

where the logarithm has its principal value. The condition $f(z) \neq 0$ for $|z| < R$ ensures that $F(z)$ is regular for $|z| < R$. Further, for $|z| \leqq r < R$,

$$\Re F(z) = \log |f(z)| \leqq \log M(r).$$

The hypothesis that either $f'(0) \neq 0$, or $f^{(k)}(0) \neq 0$ with $f^{(n)}(0) = 0$ for $n \leqq k-1$, $k \geqq 2$, gives

$$f(z) = f(0) + \sum_{n=k}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \qquad k \geqq 1,$$

or

$$F(z) = \log f(z) = \log f(0) + \sum_{n=k}^{2k-1} \frac{1}{n!} \frac{f^{(n)}(0)}{f(0)} z^n + \sum_{n=2k}^{\infty} a_n z^n, \qquad k \geqq 1,$$

for $|z| < R$, $\log f(z)$ being analytic in this region. Hence, equating the coefficients of z^n, $k \leqq n \leqq 2k-1$, in the power series for $F(z)$ in $|z| < R$ and in the above representation of $F(z)$, we get

$$F^{(n)}(0) = \frac{f^{(n)}(0)}{f(0)}, \qquad 1 \leqq k \leqq n \leqq 2k-1.$$

Consequently we can replace $f(z)$ by $F(z)$ in (1) and obtain, for $k \leqq n \leqq 2k-1$,

$$\frac{1}{n!} \left| \frac{f^{(n)}(0)}{f(0)} \right| = \frac{|F^{(n)}(0)|}{n!} \leqq \frac{2}{R^n} [\log M(R) - \log |f(0)|],$$

i.e.,

$$\frac{|f^{(n)}(0)|}{n!} \leqq \frac{2}{R^n} \left[|f(0)| \log \frac{M(R)}{|f(0)|} \right].$$

When the expression within square brackets in the right-hand member of the last inequality is replaced by its absolute maximum which corresponds to $|f(0)| = M(R)/e$, the inequality becomes identical with (3).

(3′) may be derived from (1′) in the same way as (3) from (1). Alternatively, (3′) may be deduced from (3) by arguing as follows. The function

$$g(z) = f\left[\frac{R^2(z+z_0)}{R^2 + \bar{z}_0 z}\right], \qquad |z_0| < R,$$

satisfies the same conditions as $f(z)$. Hence (3) holds for $g(z)$ with $n=1$, *i.e.*,

$$|g'(0)| \leqq \frac{2}{R} \frac{M(R)}{e}, \quad \text{or} \quad |f'(z_0)| \frac{R^2 - |z_0|^2}{R^2} \leqq \frac{2}{R} \frac{M(R)}{e}.$$

The last inequality is the same as (3′) with z_0 in place of z.

REMARKS ON (3), (3′). *The inequality* (3) *cannot be improved by replacing its right-hand member by a smaller number.* For, in the example

$$f(z) = \exp\left(\frac{z^k + 1}{z^k - 1}\right), \qquad R = 1, \tag{4}$$

where k is a positive integer, we have $f^{(n)}(0) = 0$ for $n \leqq k-1$ when $k > 1$, and

$$\frac{|f^{(k)}(0)|}{k!} = \frac{2}{e} = \frac{2}{R^k}\,\frac{M(R)}{e}$$

whether $k > 1$ or $k = 1$.

(3′) *again cannot be improved to*

$$|f'(z)| \leqq \frac{2R}{R^2 - r^2}\left(\frac{M}{e} - \delta\right), \qquad \delta > 0.$$

For, when $f(z)$ is the function in (4), with $k = 1$, the left-hand member of the above inequality tends to $2M/eR$ as $r \to 0$, while the right-hand member tends to $2(M/e - \delta)/R$, thus leading to a contradiction for all z sufficiently close to 0.

It may also be pointed out in regard to (3′) that its right-hand member must contain a factor such as $1/(R-r)$ which tends to infinity as $r \to R$. A consideration of the function $f(z)$ in (4), with $k = 1$, is sufficient to show this.

An example to show that (3) *is not necessarily true when* $f(z)$ *vanishes in* $|z| < R$ is furnished by the function

$$f(z) = \frac{z^k - \alpha}{1 - \alpha z^k}, \qquad 0 < \alpha < \sqrt{1 - \frac{2}{e}}, \qquad R = 1, \tag{5}$$

where k is a positive integer. This function is such that $f^{(n)}(0) = 0$ for $n \leqq k-1$ when $k > 1$; and, whether $k > 1$ or $k = 1$,

$$\frac{|f^{(k)}(0)|}{k!} = 1 - \alpha^2 > \frac{2}{e} = \frac{2}{e}\,\frac{M(R)}{R^k}.$$

The function $f(z)$ in (5) shows also that (3′) *is not always true when* $f(z)$ *vanishes in* $|z| < R$.

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A REMARK ON FINITE SIMPLE RINGS

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In his paper, *Two element generation of a separable algebra* [1], A. A. Albert has proved that, if A is a separable associative algebra of finite dimension over an *infinite* field F, then A can be generated over F by one or two elements according as A is or is not commutative. The restriction on F is necessary, since a diagonal algebra A of dimension n over a finite field F of r elements cannot be generated by a single element in case $n>r$; t elements are required to generate a diagonal algebra of dimension $n=rt$. Similarly, at least t elements are needed to generate a noncommutative algebra over F which is a direct sum of ideals, one of which is a diagonal algebra of dimension $n=rt$.

However, if we restrict our attention in the case of a finite field F to simple algebras, we obtain Albert's conclusion.‡ Stating our theorem for rings is a slight generalization, since the coefficients are then ordinary integers instead of elements of F.

THEOREM. *Any finite simple associative ring A can be generated by one or two elements according as A is or is not commutative.*

Proof. If A is commutative, then A is a finite field, and the set A^* of nonzero elements is a multiplicative cyclic group. If A is not commutative, then A is the ring of all $m\times m$ matrices ($m\geqq 2$) with elements in a finite division ring F (necessarily a field). Let

$$y=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\cdot&\cdot&\cdot&&\cdot\\\cdot&\cdot&\cdot&&\cdot\\\cdot&\cdot&\cdot&&\cdot\\0&0&0&\cdots&1\\1&0&0&\cdots&0\end{pmatrix}$$

in A. Then $y^m=1$. Writing e_{ij} for the matrix with 1 at the intersection of the ith row and jth column, zeros elsewhere, it is easy to see that

$$y^{m-i+1}e_{11}=e_{i1},\qquad e_{11}y^{j-1}=e_{1j},$$

so that, as in [2, p. 92, formula (49)], we have

$$e_{ij}=e_{i1}e_{1j}=y^{m-i+1}e_{11}y^{j-1}. \tag{1}$$

Hence any element of A may be written in the form

$$a=\sum\alpha_{ij}e_{ij}=\sum y^{m-i+1}(\alpha_{ij}e_{11})y^{j-1} \tag{2}$$

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‡ Actually, in the case of two element generation, Albert proves more; namely, that any element may be written in the special form $\sum\alpha_{ij}x^iy^j$, but this is not the essential part of the theorem.

for α_{ij} in F. Let θ be a generator of the multiplicative cyclic group F^*. Then any $\alpha_{ij} \neq 0$ may be written as $\alpha_{ij} = \theta^{k_{ij}}$ so that

$$\alpha_{ij} e_{11} = (\theta e_{11})^{k_{ij}}, \qquad \alpha_{ij} \neq 0 \text{ in } F. \tag{3}$$

Combining (2) and (3), we have either $a = 0$ or (by suppressing all zero terms)

$$a = \sum y^{m-i+1} (\theta e_{11})^{k_{ij}} y^{j-1}.$$

That is, y and θe_{11} generate A.

The referee points out that if B is any ring with unity element, then formula (1) holds for the matrix units e_{ij} of the ring B_m of all $m \times m$ matrices with elements in B. This implies that if B is generated by a subset S, then B_m is generated by y and the elements of $e_{11}S$ (since e_{11} itself is generated by the elements of $e_{11}S$). Our theorem is a corollary of this result.

References

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A THEOREM ON A SPECIAL CLASS OF NEAR-VANDERMONDE DETERMINANTS

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Let us call a determinant of the form

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ x_1^4 & x_2^4 & \cdots & x_n^4 \\ x_1^6 & x_2^6 & \cdots & x_n^6 \\ x_1^{\cdot} & x_2^{\cdot} & \cdots & x_n^{\cdot} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_1^{2(n-1)} & x_2^{2(n-1)} & \cdots & x_n^{2(n-1)} \end{vmatrix} \tag{1}$$

a near-Vandermonde determinant. The motivation for this definition arises from the fact that when $x_1 x_2 \cdots x_n$ is factored out of the above determinant, what is left is nearly a Vandermonde determinant.

Special classes of Vandermonde and near-Vandermonde determinants appear in a natural manner in the study of trigonometrical series. (See for example Hardy [1, p. 31], where it is shown how Fourier in his work was led to the study of systems of equations whose determinants are essentially Vandermonde.) In the author's study of localization phenomenon in double trigonometric series, Shapiro [2], a special class of near-Vandermonde determinants arose quite naturally, namely the class of determinants

$$\Delta_n = \begin{vmatrix} 1 & 2 & \cdots & (n-1) & n \\ 1 & 2^2 & \cdots & (n-1)^2 & n^2 \\ 1 & 2^4 & \cdots & (n-1)^4 & n^4 \\ 1 & 2^6 & \cdots & (n-1)^6 & n^6 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & 2^{2(n-1)} & \cdots & (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix}$$

which occurs when $x_i = i$ in (1). It became necessary in the above-mentioned work to show that there is an infinite number of n for which $\Delta_n \neq 0$, which is the theorem to be proved in this note.

THEOREM. $\Delta_n \neq 0$ *for an infinite number of* n.

Let p be any odd prime number. Set $n = \frac{1}{2}(p+1)$. It will be shown that for n so chosen, $\Delta_n \neq 0$. The first row of Δ_n can be rewritten as

$$n - (n-1) \qquad n - (n-2) \cdots n - 1 \qquad n - 0$$

and consequently we see that $\Delta_n = A - B$, where

$$A = n \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2^2 & \cdots & (n-1)^2 & n^2 \\ 1 & 2^4 & \cdots & (n-1)^4 & n^4 \\ 1 & 2^6 & \cdots & (n-1)^6 & n^6 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & 2^{2(n-1)} & \cdots & (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix} \tag{2}$$

and

$$B = \begin{vmatrix} n-1 & n-2 & \cdots & 1 & 0 \\ 1 & 2^2 & \cdots & (n-1)^2 & n^2 \\ 1 & 2^4 & \cdots & (n-1)^4 & n^4 \\ 1 & 2^6 & \cdots & (n-1)^6 & n^6 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & 2^{2(n-1)} & \cdots & (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix}. \tag{3}$$

Now the determinant in (2) is a special case of the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_1^2 & x_2^2 & \cdots & x_{n-1}^2 & x_n^2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} \end{vmatrix} = \prod_{i<j} (x_j - x_i) \qquad \begin{matrix} i = 1, \cdots, n-1 \\ j = 2, \cdots, n \end{matrix}$$

where $x_i = i^2$ for $i = 1, \cdots, n$. Consequently A contains as a factor $n^2-(n-1)^2 = 2n-1$. But $2n-1 = p$. Therefore p divides A.

Expanding B by means of the first row, we see that the cofactor of each element in this row is a Vandermonde determinant of order $n-1$, multiplied by the square of constants which are less than or equal to n^2. Thus, for example the cofactor of $n-1$ is

$$\begin{vmatrix} 2^2 & \cdots & (n-1)^2 & n^2 \\ 2^4 & \cdots & (n-1)^4 & n^4 \\ 2^6 & \cdots & (n-1)^6 & n^6 \\ \vdots & & \vdots & \vdots \\ 2^{2(n-1)} & \cdots & (n-1)^{2(n-1)} & n^{2(n-1)} \end{vmatrix}$$

(4)

$$= 2^2 \cdots (n-1)^2 n^2 \begin{vmatrix} 1 & \cdots & 1 & 1 \\ 2^2 & \cdots & (n-1)^2 & n^2 \\ 2^4 & \cdots & (n-1)^4 & n^4 \\ \vdots & & \vdots & \vdots \\ 2^{2(n-2)} & \cdots & (n-1)^{2(n-2)} & n^{2(n-2)} \end{vmatrix}.$$

The determinant on the right side of (4) contains as a factor $n^2-(n-1)^2 = 2n-1 = p$. Therefore p divides the cofactor of $n-1$. Likewise p divides the cofactor of $n-2$, $n-3$, $\cdots$, 2. 0 multiplied by its cofactor gives a number divisible by p. So in order to show that p does not divide B, it only remains to show that p does not divide the cofactor of 1. But the cofactor of 1 is

$$(-1)^n 1 \cdot 2^2 \cdots (n-2)^2 n^2 \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2^2 & \cdots & (n-2)^2 & n^2 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2^{2(n-2)} & \cdots & (n-2)^{2(n-2)} & n^{2(n-2)} \end{vmatrix}$$

(5)

$$= (-1)^n 1 \cdot 2^2 \cdots (n-2)^2 n^2 \prod_{i<j} (j^2 - i^2)$$

$$i = 1, \cdots, n-2; \; j = 2, \cdots, n-2, n.$$

Noticing that $\prod_{i<j}(j^2-i^2) = \prod_{i<j}(j-i)(j+i)$ where $j-i \leqq n-1$ and $j+i \leqq 2n-2$, we conclude that p does not divide the right side of (5) and consequently that p does not divide B. On the other hand we have already shown that p does divide A. Therefore Δ_n cannot be equal to zero; which concludes the proof of the theorem.

References

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NOTE ON ONE TYPE OF INDETERMINATE FORM

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In what follows, we shall try to obtain an expression for $\lim_{n\to\infty}\{[\phi(n)]^{n^{-r}}\}$, where $\phi(n)\geqq 0$ and $\phi(\infty)=\infty$. In case $r=1$, use is made of the series $\sum_{n=0}^{\infty}\phi(n)z^n$ and its radius of convergence ρ. For on the one hand $1/\rho=\lim_{n\to\infty}\{\phi(n)^{1/n}\}$ and on the other $1/\rho=\lim_{n\to\infty}\{\phi(n)/\phi(n-1)\}$ when this limit exists. When both limits exist, they are equal and $\lim_{n\to\infty}\{\phi(n)^{1/n}\}=\lim_{n\to\infty}\{\phi(n)/\phi(n-1)\}$. For $r>1$, we proceed as follows:

Let $\lim_{n\to\infty}A_n^{(0)}$ exist and equal A. Define $A_n^{(r)}$ by

$$A_n^{(r)}=\sum_{p=1}^{n}A_p^{(r-1)}.$$

We shall now prove:

THEOREM I.

$$\lim_{n\to\infty}\frac{A_n^{(r)}}{n^r}=\frac{A}{r!}.$$

Define $E_1(s)=s$, $E_r(s)=\sum_{p=1}^{s}E_{r-1}(p)$. A simple induction proves that

$$E_r(s)=\binom{s+r-1}{r}$$

and therefore for a fixed r, $\lim_{s\to\infty}(E_r(s)/s^r)=1/r!$. $A_n^{(r)}$ can be written as $\sum_{p=1}^{n}E_{r-1}(n-p+1)A_p^{(0)}$. Choose n_0 such that for all positive k, $|A-A_{n_0+k}^{(0)}|<\epsilon$, where ϵ is an arbitrary positive number. Our theorem states that

$$\left|\frac{A}{r!}-\frac{A_n^{(r)}}{n^r}\right| \quad \text{or} \quad \left|\frac{A}{r!}-\frac{\sum_{p=1}^{n}E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right|$$

can be made arbitrarily small by choosing n large enough. Let

$$\Gamma_{n,n_0}=\left|\frac{\sum_{p=1}^{n_0}E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right| \quad \text{and}$$

$$\Delta_{n,n_0}=\left|\frac{A}{r!}-\frac{\sum_{p=n_0+1}^{n}E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right|.$$

It will be sufficient to prove that $\Gamma_{n,n_0}+\Delta_{n,n_0}$ can be made arbitrarily small. Now

$$\Delta_{n,n_0} = \left|\left(\frac{A}{r!} - \frac{A}{r!}\left(\frac{n-n_0}{n}\right)^r\right) + \left(\frac{A}{r!}\left(\frac{n-n_0}{n}\right)^r - \frac{AE_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r\right)\right.$$

$$\left. + \left(\frac{AE_r(n-n_0)}{n^r} - \frac{\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right)\right|.$$

Therefore Δ_{n,n_0} is less than or equal to the sum of the absolute values of the three terms. This may be written as

$$\begin{aligned}
\Delta_{n,n_0} &\leqq A\left\{\left|\frac{1}{r!} - \frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r\right|\right. \\
&\quad \left. + \left|\frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r - \frac{E_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r\right|\right\} \\
&\quad + \left|\frac{\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1)A}{n^r} - \frac{\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1)A_p^{(0)}}{n^r}\right| \\
&\leqq A\left\{\left|\frac{1}{r!} - \frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r\right|\right. \\
&\quad \left. + \left|\frac{1}{r!}\left(\frac{n-n_0}{n}\right)^r - \frac{E_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r\right|\right\} \\
&\quad + \left|\frac{E_r(n-n_0)}{(n-n_0)^r}\left(\frac{n-n_0}{n}\right)^r \epsilon\right|
\end{aligned}$$

since $|A - A_p^{(0)}| < \epsilon$ for $p > n_0$ and $\sum_{p=n_0+1}^{n} E_{r-1}(n-p+1) = E_r(n-n_0)$. Using the fact that $\lim_{n\to\infty}((n-n_0)/n)^r = 1$ and $\lim_{s\to\infty}(E_r(s)/s^r) = 1/r!$ we find that for a fixed n_0, n may be chosen so large that Δ_{n,n_0} differs from $\epsilon/r!$ by less than an arbitrary η. At the same time we can choose

$$n > \left[n_0 \sum_{p=1}^{n_0} E_{r-1}(n-p+1)A_p^{(0)}\right]^{1/r}.$$

Then $\Gamma_{n,n_0} < 1/n_0$ and $\Gamma_{n,n_0} + \Delta_{n,n_0} < 1/n_0 + \epsilon/r! + \eta$. By first choosing n_0 sufficiently large $1/n_0 + \epsilon/r!$ can be made arbitrarily small, and then by choosing n large enough with respect to n_0, η can be made arbitrarily small. Theorem I results immediately.

We may now proceed to evaluate $\lim_{n\to\infty}\{[\phi(n)]^{n^{-r}}\}$. Let this limit exist and equal x. Then

$$\log x = \lim_{n\to\infty}\left\{\frac{\log \phi(n)}{n^r}\right\}.$$

If, now, $\log \phi(n)$ can be written in the form $A_n^{(r)}$, where $A_n^{(r)}$ is defined as before, then

$$\log x = \lim_{n\to\infty} \frac{A_n^{(r)}}{n^r} = \frac{A}{r!} = \frac{\lim_{n\to\infty} A_n^{(0)}}{r!}$$

by Theorem I. Now, $A_n^{(r-1)} = A_n^{(r)} - A_{n-1}^{(r)}$. Apply this to $A_n^{(r)} = \log \phi(n)$ to obtain

$$A_n^{(r-1)} = \log \{ [\phi(n)][\phi(n-1)]^{-1} \}.$$

Similarly

$$A_n^{(r-2)} = \log \{ [\phi(n)][\phi(n-1)]^{-2}[\phi(n-2)] \}.$$

We may now prove by induction that

$$A_n^{(0)} = A_n^{(r-r)} = \log \{ [\phi(n)]^{\binom{r}{0}} [\phi(n-1)]^{-\binom{r}{1}} [\phi(n-2)]^{\binom{r}{2}} \cdots [\phi(n-r)]^{(-1)^r \binom{r}{r}} \}.$$

The result is

THEOREM II. *When the limit on the right exists,*

$$\lim_{n\to\infty} \{ [\phi(n)]^{n^{-r}} \} = \lim_{n\to\infty} \{ [\phi(n)]^{\binom{r}{0}} [\phi(n-1)]^{-\binom{r}{1}} [\phi(n-2)]^{\binom{r}{2}} \cdots [\phi(n-r)]^{(-1)^r \binom{r}{r}} \}^{1/r!}. \tag{1}$$

The following example will make clear the steps taken. Let

$$A_n^{(0)} = \log \left(1 + \frac{1}{n}\right); \qquad A_n^{(1)} = \sum_{p=1}^{n} A_p^{(0)} = \log (n+1);$$

$$A_n^{(2)} = \sum_{p=1}^{n} A_p^{(1)} = \log (n+1)!.$$

Theorem I states that

$$\lim_{n\to\infty} \left\{ \frac{\log (n+1)!}{n^2} \right\} = \lim_{n\to\infty} \frac{A_n^{(2)}}{n^2} = \frac{\lim_{n\to\infty} A_n^{(0)}}{2!} = 0.$$

Here $\phi(n) = (n+1)!$. Hence $\lim_{n\to\infty} [(n+1)!]^{n^{-2}} = 1$. This result would have been obtained by substituting in (1). Naturally in this case we could have applied l'Hospital's rule, writing $(n+1)! = \Gamma(n+2)$. The following is an example of a case in which l'Hospital's rule would be inapplicable. Let it be required to find $\lim_{n\to\infty} \{ [\phi(n)]^{n^{-2}} \}$ where $\phi(n)$ is defined as a function satisfying

$$\phi(n) = \phi(n-1)\left\{\prod_{\nu=1}^{n}\left(\frac{\nu+2}{\nu}\right)^{\nu}\right\}.$$

To solve this we write

$$\frac{\phi(n)}{\phi(n-1)} = \prod_{\nu=1}^{n}\left(\frac{\nu+2}{\nu}\right)^{\nu}; \qquad \frac{\phi(n-1)}{\phi(n-2)} = \prod_{\nu=1}^{n-1}\left(\frac{\nu+2}{\nu}\right)^{\nu}.$$

From (1),

$$\lim_{n\to\infty}[\phi(n)]^{n^{-2}} = \lim_{n\to\infty}\left\{\frac{\left[\frac{\phi(n)}{\phi(n-1)}\right]}{\left[\frac{\phi(n-1)}{\phi(n-2)}\right]}\right\} = \lim_{n\to\infty}\left(\frac{n+2}{n}\right)^{n/2}$$

$$= \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n/2} = e.$$

NOTE ON FINDING THE INTEREST RATE

H. E. STELSON, Michigan State College

In this note some formulas are presented for very accurate determination of the rate of interest in an annuity or installment payment plan. Consider the ordinary annuity formula

$$B = Ra_{\overline{n}|} = R\frac{1-(1+r)^{-n}}{r} \tag{1}$$

where

R = periodic payment*

n = number of periodic payments

B = unpaid balance at the beginning of the credit period

r = periodic rate (unknown).

Now formula (1) can be written in the form

$$B = \frac{I}{\frac{nr}{1-(1+r)^{-n}} - 1} \tag{2}$$

where

$I = Rn - B$, the cost of the loan.

The right member of (2) may be expanded in a series as follows:

* The rate of interest in installment payment plans, by the author, this MONTHLY, vol. 56, 1949, pp. 257–261.

$$B = \frac{2I}{r(n+1)}\left[1 - \frac{n-1}{6}\,r + \frac{(n-1)(n+2)}{36}\,r^2 - \frac{(n-1)(n+2)(n+8)}{540}\,r^3 + \cdots\right]. \tag{3}$$

We now invert series (3) and obtain†

$$r = \frac{2M}{n+1}\left[1 - \frac{(n-1)}{3(n+1)}\,M + \frac{(n-1)(2n+1)M^2}{9(n+1)^2} - \frac{(n-1)(2n+1)(2.2n+1.4)}{27(n+1)^3}\,M^3 + \cdots\right] \tag{4}$$

where

$$M = \frac{I}{B} = \frac{Rn - B}{B}.$$

Now (4) may be expressed as a continued fraction as follows:

$$r = \frac{2M}{n+1}\left[\cfrac{1}{1 + \cfrac{\frac{(n-1)}{3(n+1)}\,M}{1 + \cfrac{\frac{n+2}{3(n+1)}\,M}{1 + \cfrac{\frac{2n+1}{15(n+1)}\,M}{\cdots}}}}\right]‡.$$

By considering the convergents we obtain some very excellent approximations for the rate, r:

$$r_1 = \frac{6M}{3(n+1) + M(n-1)}\,§ \tag{5}$$

with an error

$$\epsilon_1 < \frac{2(n-1)(n+2)M^3}{9(n+1)^3}$$

† See Shurig's and Baily's Formulae for Finding the Interest Rate, R. W. Snyder, The American Accountant, December, 1932. In this paper Shurig's formula $r = 2M[1+M]^{-(n-1)/3(n+1)}/(n+1)$ is derived by examination of series (4). Shurig's formula, which should be better known, gives about the same accuracy as r_1 given later in this note.

‡ If the quantity in the brackets is taken as 1, we have the Constant Ratio Formula.

§ The first convergent for r gives the same formula as derived in the paper in this MONTHLY See first footnote, p. 703.

$$r_2 = \frac{2M}{n+1}\left[\frac{1+\dfrac{n+2}{3(n+1)}M}{1+\dfrac{(2n+1)}{3(n+1)}M}\right] \tag{6}$$

with an error

$$\epsilon_2 < \frac{2(n-1)(2n+1)(n+2)M^4}{135(n+1)^4}$$

$$r_3 = \frac{2M}{n+1}\left[\frac{1+\dfrac{7n+11}{15(n+1)}M}{1+\dfrac{4n+2}{5(n+1)}M+\dfrac{(n-1)(2n+1)}{45(n+1)^2}M^2}\right]. \tag{7}$$

This value of r_3 expanded in a series is the same as the series for r for the first four terms. Hence the error ϵ_3 must be the order of M^5/n.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

SIMPLE PROBLEMS ON ARC LENGTH

H. S. THURSTON, University of Alabama

The author of a popular text-book* in elementary calculus uses the determination of the circumference of a circle as an illustration of the use of the formula $s=\int\sqrt{1+y'^2}$. In a footnote he points out that a vicious circle is involved (although it develops that the girth of a vicious circle is the same as that of its gentle and saddle-broken cousin, *viz.* $c=2\pi r$) since it was necessary to know in advance that the circumference is $2\pi a$ in order to evaluate arcsin 1 which arises in the solution. He adds that the problem was chosen as an illustration "in order not to diminish the limited supply of simple problems."

A question arises as to the significance of the word "simple." One can easily infer that there is only a finite number of curves the determination of whose arc length is within the capabilities of the college student. On the other hand, the author may mean that there is a limited supply of problems which the average student can work in, let us say, fifteen minutes per problem. While the

* Love, Differential and Integral Calculus, 4th edition, Macmillan, 1943.

writer is inclined to agree with the latter interpretation, he wishes to point out that there is an unlimited supply of problems on this topic which involve only the usual techniques of integration.

It will be conceded, I am sure, that a simplification is achieved if $1+y'^2$ is a perfect square. Such functions as $y=\log\cos x$ and $y=\cosh x$ lend themselves to this type of simplification because of the identities $1+\tan^2 x=\sec^2 x$ and $1+\sinh^2 x=\cosh^2 x$. Problems involving the arc lengths of these curves are found in nearly every text-book in elementary calculus. The functions may be regarded as special cases of an infinite set of functions now to be considered.

Let $y=F(x)+G(x)$. Then $1+y'^2=1+F'^2+2F'G'+G'^2$. If we choose as F' and G' functions whose indefinite integrals can be found and such that $4F'G'+1=0$, then $1+y'^2=(F'-G')^2$ and $s=\int_a^b\sqrt{1+y'^2}=\int_a^b|F'(x)-G'(x)|\,dx$. The infinite variety of choices for F' and G' may well furnish both authors and instructors with a refreshingly new set of problems on the topic under consideration.

As noted above $y=\log\cos x$ and $y=\cosh x$ may be regarded as functions of the type just indicated. The definition of $\cosh x$ represents this function in the form $F(x)+G(x)$ with $4F'G'+1=0$, whereas $y=\log\cos x$ may be written as $y=\log(\cos\frac{1}{2}x-\sin\frac{1}{2}x)+\log(\cos\frac{1}{2}x+\sin\frac{1}{2}x)=F(x)+G(x)$, the condition involving F' and G' again being satisfied.

Of special interest, if simplicity is desired, are those functions for which $F'=ax^m$ and $G'=bx^n$. On imposing the condition $4F'G'+1=0$ we are led to the equations $m+n=0$, $4ab+1=0$, whence, if m is neither 1 nor -1,

$$y=\frac{ax^{m+1}}{m+1}+\frac{x^{1-m}}{4a(m-1)} \tag{1}$$

and

$$y=\frac{ax^2}{2}-\frac{1}{4a}\log x, \qquad y=a\log x-\frac{1}{8a}x^2 \tag{2}$$

for $m=1$ and $m=-1$, respectively.

In preparing a set of problems for classroom use the complexity of the functions may vary from one as simple as $y=3x/4$ (by putting $a=1$ and $m=0$ in (1)) to something of the type

$$y=1/4\left[\left(\log\frac{x-2}{\sqrt{x^2+4}}+\arctan\tfrac{1}{2}x\right)+(8\log x-4x+x^2-x^3/3)\right]$$

constructed from

$$F'=\frac{x}{(x-2)(x^2+4)}, \qquad G'=\frac{-(x-2)(x^2+4)}{4x}, \qquad x>2.$$

It is readily seen that there is no limit to the complexity of the horrible monstrosities which may be constructed after this pattern, and no end to the supply.

VECTOR ANALYSIS BEFORE SOLID ANALYTIC GEOMETRY

O. J. KARST, JR., Stevens Institute of Technology

Vector Analysis is a concise, powerful tool which illuminates three dimensional geometry with a notation and formulation both descriptive and simple. Furthermore it is now found in the undergraduate literature with increasing frequency, while on the graduate level it is an assumed part of the student's mathematical repertoire. The earliest opportunity to introduce the elementary concepts of vector analysis should be taken.

One way in which this may be done is in connection with the presentation of solid analytic geometry. The author in a course recently taught to sophomores in an engineering college spent eight hours preceding the work in solid analytic geometry on an introduction to vector analysis. The material covered was the basic ideas of vector algebra, omitting any reference to vector calculus. A detailed outline will be given below.

With even this very brief introduction to vector analysis as a forerunner, it was then possible to attack the solid analytic geometry with the advantage of prior knowledge of unit vectors, direction cosines, direction angles, dot and cross product, and manipulative skill with vectors expressed in terms of rectangular components.

The strategic advantage thus held over the theorems and problems of solid analytic geometry was most decisive. Proofs and solutions that would have been complicated (but not profound) became elegant exercises in the vector technique. Moreover, the student kept always clearly before him the geometrical perception of the work.

In general, the author felt that the procedure was successful. The required work in solid analytic geometry was covered with what was unquestionably a superior educational approach. The rote memory work was kept at a minimum and the emphasis was on geometrical analysis. The vector technique also reduced considerably the algebraic and numerical computations attendant on the usual presentation of solid analytic geometry.

In conclusion here is presented an outline of the specific topics covered in vector analysis during eight hours of instruction.

1. Review of vector addition using triangle, parallelogram, polygon, and component methods.
2. Direction cosines and their relation to rectangular components.
3. The general unit vector and the fact that its components are the direction cosines.
4. Unit vectors along the coordinate axes and the representation of general vectors in the rectangular component form.
5. The dot product:
   a. Definition
   b. Development of formula based on rectangular components
   c. Application to physics.

6. The cross product:
   a. Definition
   b. Development of formula based on rectangular components expressed in form of a determinant. (This affords an opportunity for a brief review of determinant theory.)
   c. Application to physics.

AN APPLICATION OF THE VANDERMONDE DETERMINANT

HARLEY FLANDERS, University of California, Berkeley

Let $x_1, \cdots, x_n$ be n numbers such that $\sum x_i = \sum x_i^2 = \cdots = \sum x_i^n = 0$; to prove that $x_1 = x_2 = \cdots = 0$. This is usually done by means of Newton's identities which, however, are not completely elementary. Here is a proof based on the Vandermonde determinant and the fact that a determinant vanishes as soon as its rows are linearly dependent.

Set

$$V(y_1, \cdots, y_n) = \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} \end{vmatrix}.$$

Then we know that $V(y)$ vanishes if and only if $y_i = y_j$ for some $i \neq j$. The assumed relations imply that the rows of the determinant

$$\begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^n \\ \cdot & \cdot & \cdots & \cdot \\ x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

are linearly dependent; hence it vanishes. But this determinant is precisely $x_1 \cdots x_n \cdot V(x_1, \cdots, x_n)$. If some of the x_i are equal to zero, then the same conditions hold, simply with a smaller value of n. We may assume none of the x_i vanish and conclude that $V(x_1, \cdots, x_n) = 0$, and hence two of the x_i are equal. Assume, without loss of generality, that $x_{n-1} = x_n$. The conditions become $x_1 + \cdots + x_{n-2} + 2x_{n-1} = 0$, $x_1^2 + \cdots + x_{n-2}^2 + 2x_{n-1}^2 = 0$, *etc.* Thus the rows of the determinant

$$\begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{vmatrix}$$

are linearly dependent; hence it vanishes, and so we conclude that $V(x_1, \cdots, x_{n-1}) = 0$, $x_{n-1} = x_{n-2}$, and finally, repeating this, $x_1 = \cdots = x_n$. Since $\sum x_i = 0$, we have $nx_1 = 0$, $x_1 = 0$, $x_1 = x_2 = \cdots = x_n = 0$.

AN EVERYWHERE CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTION

JOHN MCCARTHY, Princeton University

The following is an especially simple example. It is

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g(2^{2^n} x)$$

where $g(x) = 1+x$ for $-2 \leqq x \leqq 0$, $g(x) = 1-x$ for $0 \leqq x \leqq 2$ and $g(x)$ has period 4.

The function $f(x)$ is continuous because it is the uniform limit of continuous functions. To show that it is not differentiable, take $\Delta x = \pm 2^{-2^k}$, choosing whichever sign makes x and $x+\Delta x$ be on the same linear segment of $g(2^{2^k}x)$. We have

1. $\Delta g(2^{2^n}x) = 0$ for $n > k$, since $g(2^{2^n}x)$ has period $4 \cdot 2^{-2^n}$.
2. $\left|\Delta g(2^{2^k}x)\right| = 1$.
3. $\left|\Delta \sum_{n=1}^{k-1} 2^{-n} g(2^{2^n}x)\right| \leqq (k-1) \max \left|\Delta g(2^{2^n}x)\right| \leqq (k-1)2^{2^{k-1}}2^{-2^k} < 2^k 2^{-2^{k-1}}$.

Hence $\left|\Delta f/\Delta x\right| \geqq 2^{-k}2^{2^k} - 2^k 2^{2^{k-1}}$ which goes to infinity with k.

The proof that the present example has the required property is simpler than that for any other example the author has seen.

Weierstrass gave the example $F(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$ for $b < 1$ and $ab > 1 + 3\pi/2$ which is discussed in Goursat-Hedrick, *Mathematical Analysis*.

A complete discussion of the construction of functions with various singular properties is given in Hobson, *Functions of a Real Variable*, volume II, Cambridge, 1926.

PREMIUM AND DISCOUNT ON BONDS

C. B. READ and J. R. HANNA, University of Wichita

In teaching the mathematics of investment it is discovered that the terms "premium on a bond" and "discount on a bond" are not defined in the same manner by various authors. Since the discrepancies involve the same concept, the discussion is restricted to "premium." Definitions found are:

I. The premium is the excess of the purchase price over the redemption value.

II. The premium is the excess of the purchase price over the face value.

An analysis of some recent books shows five using definition I, thirteen using definition II, two using the term without definition, and three using more than one definition.

[The word premium as applied to bonds is used in two senses: premium with respect to the purchase price and premium with respect to redemption price, a fact not always made clear in texts. An author may be so well aware of this that he fails to realize how lack of explicit definition may confuse the student.]

Without entering into a discussion of which definition is in use by investment firms, it seems unfortunate that authors do not mention alternative definitions. The student is justifiably confused when the author has used definition II

and then proceeds to discuss "amortization of the premium" using definition I for the "premium" without, in many cases, giving any reason for the change, or even a hint that a change has been made.

A GRAPHICAL PROCEDURE

J. P. RUSSELL, Polytechnic Institute of Brooklyn

The purpose of this note is to point out a simple graphical procedure for solving a differential equation of the following type:

$$\frac{dy}{dx} = \frac{\phi(y) + x}{\psi(x) + y}, \tag{1}$$

where $\phi(y)$ and $\psi(x)$ are continuous functions. The construction is an extension of the method of Liénard [1] used in plotting phase trajectories in non-linear problems.

One first plots the curves Γ_1: $[x = -\phi(y)]$, and Γ_2: $[y = -\psi(x)]$ as in Figure 1. Through an arbitrary point, $P(x, y)$, a line is drawn parallel to the x axis intersecting Γ_1 at A, and a line through P parallel to the y axis is drawn intersecting Γ_2 at B. One then draws a line, PD, perpendicular to AB and a small section of the integral curve of (1) passing through P is obtained by taking a small segment of PD through P. At a point close to P on PD repeat the entire procedure, and so on. The proximity of the successive points taken will determine the accuracy of the procedure.

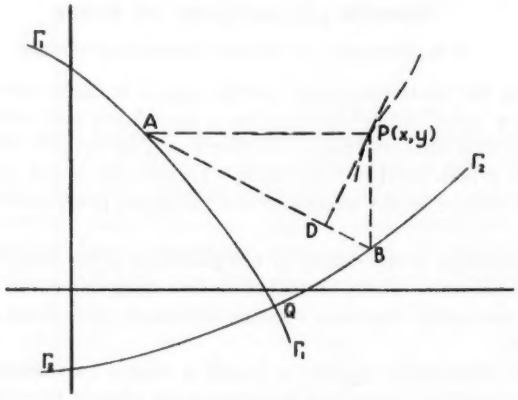


FIG. 1

The justification of the construction proceeds as follows:

The coordinates of P are $[x, y]$. Hence, the coordinates of A become $[-\phi(y), y]$; and the length of $\overline{AP}$ is therefore $\phi(y) + x$. In similar manner, the coordinates

of B are $[x, -\psi(x)]$, and hence the length of $\overline{BP}$ is $y+\psi(x)$. The slope of the line AB is $-(\psi(x)+y)/(\phi(y)+x)$, and the slope of PD is $(\phi(y)+x)/(\psi(x)+y)$, required by equation (1).

It may be remarked that the integral curves will cut Γ_1 with zero slope, and Γ_2 with vertical slope; except at Q, which we see is a singular point of the differential equation.

References

1. J. J. Stoker, Non-linear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950, pp. 31–36.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1091. *Proposed by J. Lambek, McGill University, and Leo Moser, University of Alberta*

Given a sequence of integers $n_1<n_2<\cdots<n_i<\cdots$ such that for $j>i$ the decimal representation of n_j does not begin (on the left) with the decimal representation of n_i. Prove that

$$\sum_i 1/n_i \leqq 1 + 1/2 + 1/3 + \cdots + 1/9.$$

E 1092. *Proposed by N. A. Court, University of Oklahoma*

The homothetic center of the orthic and tangential triangles of a given triangle (T) (see the proposer's *College Geometry*, 2nd ed., p. 98, art. 191) is the pole of the orthic axis of (T) with respect to the circumcircle of (T).

E 1093. *Proposed by H. S. Wilf, Nuclear Development Associates, White Plains, N. Y.*

Define $S_0=1$, $S_1=3$, $S_{n+1}=2S_n^2-1$ for $n\geqq 1$. Find

$$\lim_{n\to\infty} \frac{S_n}{2^n S_0 S_1 \cdots S_{n-1}}.$$

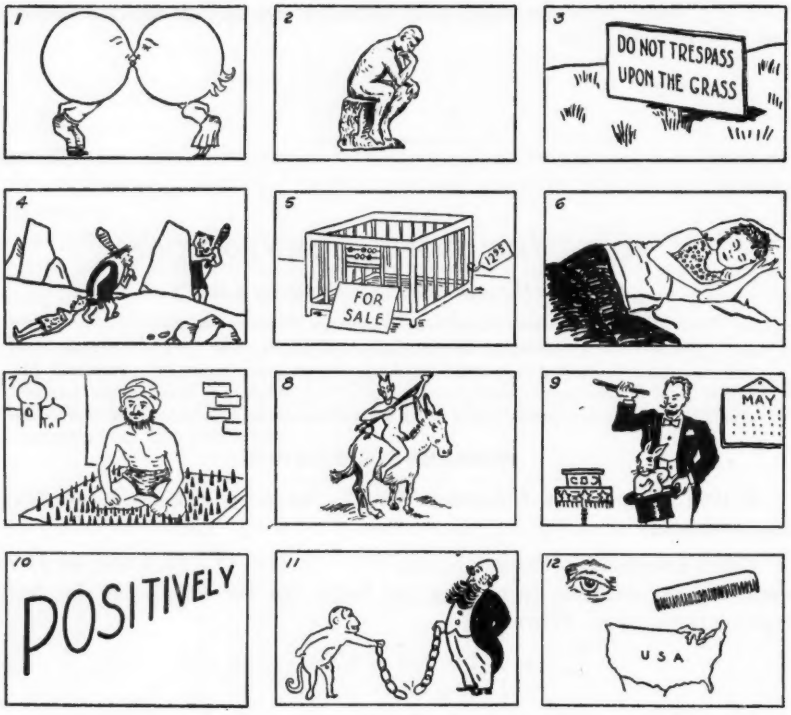
E 1094. *Proposed by Azriel Rosenfeld, Columbia University*

(1) Construct a function defined everywhere on a closed (or open) interval, which takes on each of its values exactly twice on this interval.

(2) Prove that no such function can be continuous.

E 1095. *Proposed by Leon Bankoff and C. W. Trigg, Los Angeles, Calif.*

Translate each of the following sketches into a mathematical term.



SOLUTIONS

A Quadratic Having Roots Differing by Unity

E 1061 [1953, 262]. *Proposed by Walter Penney, Washington, D. C.*

Solve for n, given that the equation

$$\sum_{i=1}^{n} (x + i - 1)(x + i) = 10n$$

has roots r and $r+1$.

Solution by Julian Braun, Washington, D. C. Expansion of the left member and further reduction yields

$$x^2 + nx + (n^2 - 31)/3 = 0, \qquad n > 0. \tag{1}$$

The sum of the roots must be $2r+1 = -n$, and the product of the roots must be $r(r+1) = (n^2-31)/3$, a simultaneous solution of which yields $n=11$ and $r=-6$.

Also solved by P. M. Anselone, J. L. Baker, W. E. Briggs, Bernard Chovitz, P. L. Chessin, J. E. Darraugh, Fred Discepoli, S. H. Eisman, Herbert Emich, D. G. Frood and D. A. Trumpler (jointly), Lloyd Fulk, Harry Furstenberg, H. M. Gehman, A. Gregory, Douglas Holdridge, P. F. Hultquist, A. R. Hyde, Herbert James, John Jones, Jr., A. E. Livingston, David Mandelbaum, R. G. McDermot, George Millman, C. S. Ogilvy, M. W. Oliphant, S. Parameswaran, W. O. Pennell, W. J. Pervin, C. F. Pinzka, Azriel Rosenfeld, C. M. Sandwick, Sr., Milton Scharf, Nathan Schwid, Michael Skalskyj, O. E. Stanaitis, A. V. Sylwester, F. Underwood, Roscoe Woods, and the proposer. Late solution by H. J. Hauer.

Frood and Trumpler found $n=11$ by using the fact that the roots of (1) will differ by unity if and only if the discriminant is equal to 1.

Fulk and Woods considered the more general problem of finding n given that

$$\sum_{i=1}^{n} (x + i - 1)(x + i) = pn$$

has roots r and $r+q$, where p and q are considered as given.

A Set of Six Positive Integers

E 1062 [1953, 262]. *Proposed by Leo Moser, University of Alberta*

(1) Find six positive integers, not exceeding 24, such that the sums of the numbers in the possible subsets of those numbers will all be different.

(2) Prove that no seven positive integers, not exceeding 24, can have sums of all subsets different.

Solution by C. F. Pinzka, Princeton, N. J. (1) Let the six integers be $n_1=a$, $n_2=a+b$, $n_3=a+b+c$, $\cdots$, $n_r=a+b+c+d+e+f$, so that a, b, c, d, e, f are all positive. If we arbitrarily assume that the "obvious" order of the subsets (*i.e.*, a sum of k numbers is greater than a sum of j numbers if $k>j$, and in two ordered sums of k numbers each that sum is greater in which one first finds a term exceeding the corresponding term in the other sum) results in an increasing sequence of sums, a set of inequalities results. The crucial inequalities (inequalities including the remaining ones) are

$$c > e+f \geqq 2,$$
$$d > f \geqq 1,$$

$$a > c + 2d + 2e + f \geqq 10,$$
$$b > d + 2e + f \geqq 5,$$

obtained from

$$n_3 + n_4 > n_2 + n_6,$$
$$n_4 + n_5 > n_3 + n_6,$$
$$n_1 + n_2 + n_3 > n_5 + n_6,$$
$$n_2 + n_3 + n_4 > n_1 + n_5 + n_6.$$

The "minimum" set $(a, b, c, d, e, f) = (11, 6, 3, 2, 1, 1)$ gives 11, 17, 20, 22, 23, 24 as a set of positive integers having the desired property. Since an arbitrary assumption was made above, there is no guarantee of uniqueness.

(2) The number of subsets of 4 or fewer numbers out of a set of 7 is (excluding the null set)

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 98.$$

If no integer in these sets exceeds 24, then 1 and 90 will be obvious lower and upper bounds for the sums of these subsets of 4 or fewer numbers. Since $90 < 98$, the sums cannot all be distinct.

Also solved by W. E. Briggs, Michael Skalskyj, and the proposer.

An Inequality

E 1063 [1953, 263]. *Proposed by J. V. Whittaker, U.C.L.A.*

Show that if $a \geqq 2$ and $x > 0$, then $a^x + a^{1/x} \leqq a^{x+1/x}$, equality holding if, and only if, $a = 2$ and $x = 1$.

Solution by A. E. Livingston, University of Washington. Let $f(x, a) = (a^x + a^{1/x})/a^{x+1/x}$ for $a \geqq 2$ and $x > 0$. We first observe that $f(x, a) = f(1/x, a)$ and, hence, that the problem will be solved if we show that $f(x, a) \leqq 1$ for $x \geqq 1$ with equality if and only if $a = 2$ and $x = 1$. We next observe that $f(x, a)$ is a strictly decreasing function of a for each x and, hence, that it is sufficient to prove that $f(x, 2) < 1$ for $x > 1$, it being clear that $f(1, 2) = 1$. We will obtain this latter inequality by showing that $F(x) = 2^x f(x, 2) - 2^x < 0$ for $x > 1$. It is easily verified that

$$F'(x) = 2^{x-1/x}(\ln 2)(1 + 1/x^2 - 2^{1/x}) = 2^{x-1/x}(\ln 2)g(1/x).$$

The proof will be complete if we can show that $g(1/x) < 0$ for $x > 1$. Now

$$g''(t) = 2 - 2^t(\ln 2)^2 > 2 - 2(\ln 2)^2 > 0$$

for $0 < t < 1$, so that $g(t)$ is strictly convex for these t. Since $g(t)$ is continuous and $g(0) = g(1) = 0$, it follows that $g(t) < 0$ for $0 < t < 1$ or that $g(1/x) < 0$ for $x > 1$.

Also solved by F. J. Duarte, A. R. Hyde, M. S. Klamkin, R. Larivière, M. J. Pascual, W. O. Pennell, O. E. Stanaitis, and the proposer.

A Converse of the Mean Value Theorem

E 1064 [1953, 263]. *Proposed by Jacob Samoloff and Albert Wilansky, Lehigh University*

Let $f(x)$ be continuous and $f'(x)$ exist in a neighborhood of $x=c$. Suppose that there exists a continuous function $\theta(h)$, with $0<\theta(h)<1$, satisfying the equation

$$f(c+h)-f(c)=hf'[c+h\theta(h)].$$

Does it follow that $f'(x)$ is continuous at $x=c$?

Solution by J. V. Whittaker, U.C.L.A. We have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0}f'(c+h\theta(h))=\lim_{q\to 0}f'(c+q),$$

for $\theta(h)$, being positive and continuous in a closed interval containing 0, must assume a positive minimum value in this interval, so that $q=h\theta(h)\to 0$ is equivalent to $h\to 0$. Therefore, $f'(x)$ is continuous at $x=c$.

Also solved by Sidney Glusman, D. S. Greenstein, Douglas Holdridge, A. E. Livingston, M. S. Klamkin, and Nathan Schwid.

Maximum Section of a Solid Right Circular Cylinder

E 1065 [1953, 263]. *Proposed by C. S. Ogilvy, Hamilton College*

Find the largest plane section of a given solid right circular cylinder.

Editorial Note. It is a mistake to assume, as did all the attempted solutions, that the maximum section is the maximum *elliptical* section. Consider, for example, a cylinder whose altitude h is equal to four times the radius r. Then the maximum elliptical section has an area $E=\sqrt{5}\pi r^2\approx 7r^2$. But the area of a rectangular section through the axis of the cylinder is $R=8r^2$.

The correct solution depends upon the fact that the maximum section of a closed convex surface having a center of symmetry must pass through the center of symmetry. V. L. Klee, Jr., has observed that this is a direct application of the Brunn-Minkowski Theorem. Therefore, the maximum section of a solid right circular cylinder must pass through the center of the cylinder, and therefore will be an ellipse, a part of an ellipse, or a rectangle. Let us consider the second type of central section. Denoting the area of the section by A and the angle the section makes with a base of the cylinder by θ, one may easily show that

$$A/2r^2=\sqrt{1+(k/u)^2}(u\sqrt{1-u^2}+\sin^{-1}u), \qquad 0<u\leqq 1,$$

where $k=h/2r$ and $u=k\cot\theta$. One may now apply standard maximum-minimum methods, but the equation $d(A/2r^2)/du=0$ probably can be solved only by approximation methods.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4563. *Proposed by Albert Edrei, Syracuse University*

Let

(1) $$\alpha_1, \alpha_2, \alpha_3, \cdots$$

be an infinite, strictly increasing sequence of positive integers, and let $B=\{b\}$ be a set of such sequences. A sequence of the form (1) is said to be *almost contained* in B if, to every b, there corresponds an integer $k=k(b)$ such that $\alpha_k, \alpha_{k+1}, \alpha_{k+2}, \cdots$ is a subsequence of b.

The following theorem is easy to prove: *Let B be denumerable and such that the intersection of any finite number of b's is an infinite sequence. Then there exists an infinite sequence almost contained in B.*

The problem is to prove that this theorem is false if the word denumerable is omitted.

4564. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Prove

$$\sum_{n=1}^{\infty} \frac{S_n}{n^3} = \frac{\pi^4}{72}, \qquad \text{where } S_n = \sum_{r=1}^{n} \frac{1}{r}.$$

4565. *Proposed by Donald Bratton, University of Chicago*

A ring A is called a valuation ring when it is commutative and, for each couple of elements (x, y) of A, either x divides y or y divides x. Show that each valuation ring has a unit element.

4566. *Proposed by C. C. Chevalley, Columbia University, and Paul Erdös, University of Notre Dame*

Show that the following relations are impossible for $n>2$:

$$\binom{2n}{n} = 2\binom{2n}{k}, \qquad \binom{2n+1}{n} = 2\binom{2n+1}{k}.$$

4567. *Proposed by R. S. Underwood, Texas Technological College*

Prove that the following simultaneous equations in the four unknowns

x, y, z, u have one and only one real solution in terms of the parameters a, b, where $0<a<5$, $0<b<5$:

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{5-a}+\frac{u^2}{5-b}=1$$

$$(x-z-4)^2+(y-u-2)^2=5.$$

SOLUTIONS

Divisors and Multiples of a Set of Integers

4352 [1949, 479]. *Proposed by Paul Erdös, University of Notre Dame*

Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(1<a_i \leqq n)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

where $2, 3, \cdots, p_k$ are the first k primes.

Solution by George Szekeres, University of Adelaide, Australia. Given n and $1 \leqq a_i \leqq n$, $i=1, \cdots, k$, let $M(n; a_1, \cdots, a_k)$ denote the set of all m, $1 \leqq m \leqq n$ for which either $a_i \mid m$ or $m \mid a_i$ for at least one a_i. Let $|M|$ denote the number of integers belonging to the set M and write

$$\lambda(a)=\alpha_1+\cdots+\alpha_r \qquad \text{if } a=p_1^{\alpha_1}\cdots p_r^{\alpha_r}.$$

Given k and n let a_i, $i=1, \cdots, k$, be a system of numbers for which $|M(n; a_1, \cdots, a_k)|$ is a maximum and (if there are several such systems) one for which $\lambda(a_1a_2\cdots a_k)$ is a minimum. We shall show that under these conditions each a_i is a prime. Clearly we may assume that $a_i \neq a_j$ if $i \neq j$.

(1) $a_i=p^\alpha$, p prime, implies $\alpha=1$.

For, if $\alpha>1$, take a system a_j' defined by $a_i'=p$, $a_j'=a_j$ for $j \neq i$. Obviously $M(n; a_1', \cdots, a_k') \supseteq M(n; a_1, \cdots, a_k)$ and $\lambda(a_1'\cdots a_k')<\lambda(a_1\cdots a_k)$.

(2) $a_i=p$, p prime, implies $(a_j, p)=1$ for $j \neq i$.

Suppose $a_j=p^\alpha a_j'$, $\alpha>0$, $(a_j', p)=1$, for some $j \neq i$. By (1) $a_j'>1$. Write $a_\nu'=a_\nu$ for $\nu \neq j$. Then as before,

$$\text{(I)} \qquad M'=M(n; a_1', \cdots, a_k') \supseteq M(n; a_1, \cdots, a_k)=M.$$

For the only elements of M which are possibly not in M' are divisible by p hence divisible by $a_i=a_i'$, hence are in M'.

(3) Let $a_i=p^\alpha d$, $(p, d)=1$, $\alpha \geqq 1$, $d>1$, then $d \nmid a_j$ for $j \neq i$.

Suppose $d \mid a_j$, $j \neq i$, and write $a_i'=p$, $a_\nu'=a_\nu$ for $\nu \neq i$. Again (I) is true since the only members of M which are possibly not in M' are divisors of d, hence of $a_j=a_j'$, hence are in M'.

(4) Suppose now that $a_1, \cdots, a_r$ are composite and $a_{r+1}, \cdots, a_k$ are primes. By (2), $(a_\mu, a_\nu)=1$, if $1 \leqq \mu \leqq r$, $r<\nu \leqq k$. Let $q_1, \cdots, q_k$ be distinct primes such that (i) q_1 is the smallest prime factor of $a_1\cdots a_r$, and (ii) every a_i is divisible

by at least one q_ν. Clearly such a system exists and we may assume that $a_\nu = q_\nu$ for $r < \nu \leqq k$. We shall show that

$$(\text{II}) \qquad |M'| = |M(n; q_1, \cdots, q_k)| \geqq |M(n; a_1, \cdots, a_k)| = |M|.$$

For $\nu = 1, \cdots, k$, let $m_\nu \geqq 1$ be the greatest divisor of a_ν which is relatively prime to $q_1 \cdots q_k$, *e.g.*, $m_\nu = 1$ for $\nu > r$. Obviously the members of M which are not in M' must be divisors of some m_ν and every $d \mid m_\nu$, $d > 1$ has this property. All we have to show is that there are at least as many members of M' not in M as the number of $d > 1$, $d \mid m_\nu$.

Suppose first that $d \mid m_\nu$, $1 < d < m_\nu$, and let α be the largest integer such that $A(d) = q_1^\alpha d \leqq n$. Clearly $\alpha \geqq 1$ and $A(d)$ is a member of M'. On the other hand $A(d)$ is not in M, *i.e.*, it is not a divisor of a_i and not divisible by a_i for any i. For $i = \nu$ this is clear, for $i \neq \nu$ it follows from (3). In fact if $A(d)$ is a divisor of a_i (*i.e.*, $i \leqq r$) then it must be equal to a_i, since $pq_1^\alpha d > n$ for any prime factor p of a_i, hence $a_i \mid A(d)$. But if a_i divides $A(d)$ then it must have the form $q_1^\beta d_1$, d_1 a divisor of d, hence of a_ν, against (3). It is also clear that for $d_1 \neq d_2$, $A(d_1) \neq A(d_2)$.

Suppose next that $d = m_\nu > 1$, *i.e.*, $\nu \leqq r$, and let their number be s. We must find s distinct numbers not exceeding n and different from all the $A(d)$ such that they are members of M' but not of M. For $i = 1, \cdots, r$, let β_i be the largest integer such that $q_i^{\beta_i} \leqq n$, and γ_i the largest integer such that $A_i = q_i^{\beta_i} q_i^{\gamma_i} \leqq n$. The numbers A_i are evidently members of M' and different from the $A(d)$. We have to show that at least s of them are not in M. Now if A_i is a divisor of a_j then it must be identical with a_j by the same argument as before, hence is divisible by a_j. Of course, it may happen that A_i is divisible by an a_j (or even several a_j) with $m_j = 1$. But two different A_i's cannot be divisible by the same a_j since this would imply that a_j is a power of q_1. It follows that there are at least as many A_i not in M as there are a_ν's with $m_\nu > 1$, *i.e.*, s. The numbers $A(d)$ and A_i together give therefore at least as many members of M', not in M, as the number of $d > 1$, $d \mid m_\nu$, *q.e.d.*

But $\lambda(q_1 \cdots q_k) < \lambda(a_1 \cdots a_k)$ if $r > 0$, hence $r = 0$ and every a_i is a prime. The remark that

$$|M(n; q_1, \cdots, q_k)| \leqq |M(n; 2, 3, \cdots, p_k)|$$

concludes the proof.

Eratosthenian Averages

4445 [1951, 422]. *Proposed by Paul Erdös, University of Notre Dame*

Split the set of primes $p_1 < p_2 < \cdots$ into two classes q_i and r_i so that $\sum 1/q_i = \sum 1/r_i = \infty$. Define $\mu'(k) = 0$ if k is a multiple of one of the r's, otherwise $\mu'(k) = \mu(k)$, where $\mu(k)$ is the Möbius symbol (0 if k has a square factor, $+1$ if k has an even number of distinct prime factors, -1 if it has an odd number). Prove that

$$\sum_{k=1}^{\infty} \frac{\mu'(k)}{k} = 0.$$

Editorial Note. After having provided an independent proof, the proposer finds the statement and proof of the problem in A. Wintner, *Eratosthenian Averages*, 1943, pp. 70–71.

Squarefull Integers

4459 [1951, 636]. *Proposed by D. J. Newman, Harvard University*

Find an asymptotic expression for the number of integers, not exceeding x, each of which has the property that each of its prime divisors divides it to the second power at least.

Editorial Note. The following solution, with a somewhat more accurate estimate, is quoted by an anonymous contributor from discussions among his colleagues.

Following Sklar [1953, 55], let us call such integers "squarefull" and denote the number of squarefull integers not exceeding x by $A(x)$. Let $c_n = 1$ if n is squarefull, $c_n = 0$ otherwise. Then $A(x) = \sum_{n \leq x} c_n$. If we set

$$f(s) = \sum_{n=1}^{\infty} c_n / n^s$$

then (see, *e.g.*, Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 2, p. 828)

$$\sum_{n \leq x} c_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)x^s}{s}\, ds. \tag{1}$$

From the values of c_n it follows that

$$f(s) = \prod_p (1 + p^{-2s} + p^{-3s} + \cdots + p^{-ks} + \cdots),$$

where the product is extended over all primes p. $f(s)$ can be written as

$$f(s) = \prod_p (1 + p^{-s}(p^s - 1)^{-1}) = \prod_p (1 - p^{-6s})/(1 - p^{-2s})(1 - p^{-3s})$$
$$= \zeta(2s)\zeta(3s)/\zeta(6s),$$

where $\zeta(s)$ stands for the Riemann ζ-function. The integral in (1) can be evaluated by Cauchy's theorem on residues and, if R_j are the residues of the integrand, by (1) and the definition of $A(x)$,

$$A(x) = \sum_j R_j. \tag{2}$$

The only singularities of the integrand are poles and the corresponding residues

can be evaluated as follows:

(a) the poles $s=1/2$ and $s=1/3$ of the numerator have the residues $Ax^{1/2}$ and $Bx^{1/3}$ respectively, with $A=\zeta(3/2)/\zeta(3)$ and $B=\zeta(2/3)/\zeta(2)$;

(b) the zero $s=0$ of the denominator leads to the residue $x^0\zeta(0)=-1/2$;

(c) $s=-n/3$, the real zeros of $\zeta(6s)$ lead to residues $a_nx^{-n/3}$; for $n=1$ in particular we have the residue $a_1x^{-1/3}$, where $a_1\neq 0$ (its value can be computed but is immaterial for our purpose);

(d) $\rho_\nu/6$ are the complex zeros of $\zeta(6s)$, with $\rho_\nu=\sigma_\nu+it_\nu$, the complex zeros of $\zeta(s)$. Correspondingly we have the residues $x^{\rho_\nu/6}R_\nu=x^{\sigma_\nu/6}\cdot x^{it_\nu/6}R_\nu$ (R_ν can be computed; if e.g., ρ_ν is a simple root, we easily obtain $R_\nu=\zeta(\rho_\nu/3)\zeta(\rho_\nu/2)/\rho_\nu\zeta'(\rho_\nu)$, etc.) If $\sigma=\varliminf \sigma_\nu$ then the sum of these residues can be written as

$$x^{\sigma/6}\sum_\nu R_\nu x^{-((\sigma-\sigma_\nu)+it_\nu)/6}=x^{\sigma/6}Q(x), \qquad |Q(x)|<\infty.$$

Adding together all residues we obtain by (2)

$$A(x)=Ax^{1/2}+Bx^{1/3}+Q(x)\cdot x^{\sigma/6}-1/2+O(x^{-1/3}), \tag{3}$$

where A and B have the indicated numerical values.

Assuming the Riemann hypothesis, $\sigma=\sigma_\nu=1/2$. Setting

$$C=\lim_{x\to\infty}\sum_\nu R_\nu x^{it_\nu/6},$$

(3) becomes $A(x)=Ax^{1/2}+Bx^{1/3}+Cx^{1/12}-1/2+O(x^{-1/3})$ if the limit defining C exists; otherwise $A(x)=Ax^{1/2}+Bx^{1/3}+O(x^{1/12})$. If we do not assume the Riemann hypothesis we still know that $\sigma<1$ and (3) becomes

$$A(x)=\{\zeta(3/2)/\zeta(3)\}x^{1/2}+\{\zeta(2/3)/\zeta(2)\}x^{1/3}+O(x^{1/6}).$$

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of The Association.

Trigonometry for Today. By M. Brooks, A. C. Schock, and A. I. Oliver. New York, Harper & Brothers, 1951. ix+200+Index and Tables, $2.96.

Plane Trigonometry. By L. M. Kells, W. F. Kern, and J. R. Bland. New York, McGraw-Hill, 1951. xi+180+Appendices, Answers, Index, Tables. $3.50.

Plane and Spherical Trigonometry. By M. Richardson, New York, Macmillan, 1950. xiv+343, Answers, Index and Tables.

Trigonometry, Plane and Spherical. By L. L. Smail, New York, McGraw-Hill, 1953. xii+295+Tables, Answers, Index. $3.75.

With the possible exception of the Brooks, Schock and Oliver, these additions to the already vast list of textbooks on trigonometry contain no really significant innovations in either subject matter or method of presentation. To be sure, each book is carefully written and will serve the purpose for which it is intended. Each has its idiosyncrasies of style which will render it more or less acceptable to the individual user. Each has an excellent format, well drawn figures, and ample exercises. Yet the fact remains that little is offered that is not already available in abundance.

The Richardson begins with a section called "A Little Logic" in which the difference between a proposition and its converse is explained. This is followed by a review of elementary algebra and geometry. The trigonometric functions are defined first for acute angles in terms of right triangles, and general triangles are solved by breaking them into right triangles. After this, coordinates are introduced, the definitions of the functions are extended, and the reduction formulas are treated. There follows a brief chapter on the law of sines, the law of cosines, and the solution of triangles without logarithms. Chapters on the addition formulas, identities, radian measure (with a note on the calculation of π), graphs, inverse functions, logarithms, and the logarithmic solution of triangles are followed by a chapter on miscellaneous topics which includes vectors, plane sailing, and small angles among others. The section on plane trigonometry closes with a chapter on complex numbers. The section on spherical trigonometry includes polar triangles, Napier's rules, the laws of sines and cosines, Napier's analogies, Delambre's formulas, and the haversine formulas. Results are applied to the terrestrial and celestial spheres.

The Smail opens with a discussion of rectangular and polar coordinates and the general definition of the trigonometric functions. However, on page 24 the point is reached where they can be redefined in terms of right triangles, and from here on the traditional mode of presentation prevails. A complete chapter on logarithms is placed at the end, and the book is arranged so that the use of logarithms can be started at any point. In almost every section there is a copious supply of exercises of varying degrees of difficulty, and these are followed by miscellaneous review exercises to the point of superfluity. There is a section on trigonometric sums, where among other items, the real and imaginary parts of a finite geometric series are summed with the aid of the addition formulas. The chapter on complex numbers closes with a section on the series representation of trigonometric functions, some remarks on the computation of tables, and a discussion of Euler's formula. The spherical trigonometry contains in addition to the topics mentioned in the discussion of the Richardson, a section on the so-called Ageton method.

In the preface to this, the third edition of Kells, Kern and Bland, the authors aver that they have made improvements "by eliminating every item that did not have a definite purpose. . . ." They have added a short chapter on vectors, and numerous new exercises. These occur especially in the chapter on graphs and the chapter on the addition formulas. The major features of the predecessors of this edition have been left intact.

In view of its elementary style, and relatively simple exercises, it would appear that the Brooks, Schock and Oliver is intended primarily for high school purposes. However, the emphasis which it places on the notion of "function" renders it rather unique among books at this level. The book begins with a chapter on coordinates, functions, and graphs. The trigonometric functions are then defined for arbitrary angles, and the Pythagorean identities are established. The graphs of the trigonometric functions are then taken up, and the student is directed to think of the functions in terms of their graphs. This point of view carries over into the chapter on reduction formulas which follows. With the exception of a chapter on the addition formulas, and a short chapter of "supplementary topics," the remainder of the book is concerned with the solution of triangles and logarithms.

H. T. MUHLY
State University of Iowa

Calculus. By C. R. Wylie, Jr. New York, McGraw-Hill Book Company, 1953. x+565 pages. $6.00.

The organization of this text differs from that of the traditional text mainly in the first chapter. Here the area in the first quadrant under the curve $y=x^2$ from $x=0$ to $x=1$ is found. After the customary use of rectangles for upper and lower approximations, the author makes use of the formula for the sum of the squares of the first n consecutive integers, which the student should already have encountered in problem lists on mathematical induction. Then by a frankly intuitive limiting process, the author finds both approximations to have the limit 1/3 and takes the area to be 1/3. After more facility is developed in finding areas by arithmetical means, the student is shown the symbol for definite integral and given a slight inkling of its further use. Also in the first chapter, the concept of slope is strongly associated with that of derivative.

Chapter two does careful work with limits and continuity, including $\lim_{\theta \to 0} (\sin \theta/\theta)$ and the number e. Chapters three and four complete the study of differentiation, covering all the traditional topics. In chapter five on integration, the definite integral is used to introduce the indefinite. Improper integrals follow immediately after a small amount of practice on proper integrals. Chapters six and seven complete the practice on formal integration. The formulas are derived with considerable rigor, but the amount of practice given the student is somewhat less than in many texts. Chapters eight, nine, and ten take care of rates, motion, and geometrical and physical applications of integration. Chapter eleven is an excellent explanation of hyperbolic functions. In chapter

twelve, analytic geometry of three dimensions follows, rather than precedes, the careful work on partial derivatives. This reviewer is glad to see the integral test of convergence included in chapter thirteen on series and to see that the comparison tests, so difficult for students to apply, are not the first tests given. Chapter fourteen includes Maclaurin's and Taylor's expansions, operations with series, and excellent material on Fourier's series. Chapters fifteen on multiple integration and sixteen on differential equations are followed by an excellent appendix containing a glossary and by tables.

The book abounds in examples, many from physics, and all completely solved. The author wisely suggests that the teacher adapt the text to his purposes by a suitable selection of examples. The examples are largely responsible for making the book so long that rather small type must be used. Further objection to the typography is the absence of sufficiently outstanding type for important statements. The individual formulas for differentiation are scattered and buried, and are not even numbered in a single sequence. The same is true for integration formulas. The reviewer has noticed several minor misprints and omissions.

The style is refreshingly direct. On page one the author admonishes the student that it is important for him to want to succeed in the study of the calculus. The excellent chapter introductions and summaries help the student to evaluate the work of each chapter and to relate it to the work as a whole. In these discussions the author takes the student into his confidence and supplies the answers to many of the student queries which most textbooks leave for the good teacher to answer.

This is a carefully written book. To use it as effectively as its own author no doubt does, a teacher would need to be in sympathy with the author's departure from the traditional course in a few places, notably in the beginning chapter and in the first chapter on integration.

ANICE SEYBOLD
North Central College

Complex Analysis. By Lars V. Ahlfors. New York, McGraw-Hill Book Company, 1953. xi+247 pp. $5.00.

Professor Ahlfors has fulfilled expectations that he would produce a scholarly and novel treatment of analytic function theory. The only disappointment which the reviewer experiences is that the author stopped too soon. One could wish for a continued discussion of more material in the same vein.

It has been generally recognized that expositions of analytic functions in English have failed to make sufficient use of relevant topological tools. This is the more surprising in view of the great mutual influence that complex variable theory and topology have exerted upon each other. In the present book Ahlfors remedies this deficiency by making systematic use of topological techniques, and he has also provided brief but reasonably adequate introductions to the topological ideas which he uses. At the same time he has retained the flavor of

classical function theory; and he has avoided the assembly-line format wherein every paragraph is labeled Theorem, Definition, Corollary, Remark, and the like. Since the book is self-contained and presupposes only a moderate mathematical maturity, it is suitable for a text in a first course in complex variables for beginning graduate students; or, since the point of view and treatment are quite different from that in standard texts, it can be used as the basis for a second course for students who already have considerable background. The reviewer tried the latter experiment, with encouraging results. The reasoning is close and condensed and requires alertness on the part of the student, but the writing and proofreading have been done with considerable care, so that a close scrutiny of details, such as places where, in the author's words, "a voluntary gap serves the purpose of saving half a page of unconstructive and dull reasoning" will be rewarding.

The most novel feature of the book is the treatment of Cauchy's theorem. The proof, which is spread over some 40 pages, interspersed with applications, introduces topology in stages. The key to the procedure is the lemma that $\int_C f(z)dz$ vanishes for every closed curve C in a region S, *regardless of the connectivity of* S, if $f(z)$ is continuous and is the derivative of an analytic function $F(z)$ in S. Since this lemma is independent of the topology of S, it is relatively easy to prove by standard methods of calculus from the Cauchy-Riemann equations for $F(z)$.

Among the numerous topics treated in the book, the following may be given special mention: Geometric properties of linear transformations, homology theory and Cauchy's formula based on the winding number of a curve, the openness of analytic mappings, normal families and the Riemann mapping theorem, the Dirichlet problem and harmonic measures, the theory of Riemann surfaces, algebraic functions, linear differential equations, and homotopy theory and the monodromy theorem.

P. W. KETCHUM
University of Illinois

NEW BOOKS RECEIVED

A First Course in Functions of a Complex Variable. By Wilfred Kaplan, Cambridge, Mass., Addison-Wesley Press, 1953. 7+134 pages. $3.50.

Foundations of Combinatorial Topology. By L. S. Pontryagin. Rochester, New York, Graylock Press, 1952. 12+99 pages. $3.00.

Three Pearls of Number Theory. By A. Y. Khinchin. Rochester, New York, Graylock Press, 1952. 64 pages. $2.00.

Foundations of the Nonlinear Theory of Elasticity. By V. V. Novozhilov. Rochester, New York. Graylock Press, 1953. 6+233 pages. $4.00.

50–100 Binomial Tables. By H. G. Romig. New York, John Wiley and Sons, Inc., 1953. 27+172 pages. $4.00.

Analytical Geometry of Three Dimensions. By W. H. McCrea. New York, Interscience Publishers, Inc., 1953. 7+144 pages. $1.25.

Squaring the Circle. By E. W. Hobson. New York, Chelsea Publishing Company, 1953. 381 pages approx. (4 books published in one). $3.25.

Mathematical Aspects of the Quantum Theory of Fields. By K. O. Friedrichs. New York, Interscience Publishers, Inc. 8+272 pages. $5.00.

Engineering Statistics and Quality Control. By I. W. Burr. New York, McGraw-Hill Book Co., Inc., 11+442 pages. $7.00.

A Refresher Course in Mathematics. By F. J. Camm. New York, Emerson Books, Inc. (251 West 19th Street, New York 11, N. Y.), 1953. 240 pages. $2.95.

Relativity and Reality. By E. G. Barter. New York, The Philosophical Library, Inc., 1953. 11+131 pages. $4.75.

Mathematics and Statistics for Economists. By Gerhard Tintner. New York, Rinehart and Company, Inc., 1953. 14+363 pages. $6.50.

Analytic and Projective Geometry. By D. J. Struik. Cambridge, Mass., Addison-Wesley Press, Inc., 1953. 9+291 pages. $6.50.

Analytische Geometrie. By Gunter Pickert. Leipzig, Germany, Akademische Verlagsgesellschaft, Geest & Portig K.-G., 1953. x+397 pages. DM 26.

Probability Tables for the Analysis of Extreme-Value Data. National Bureau of Standards Applied Mathematics Series 22, ii, 32 pages. 25 cents.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

PERSONAL ITEMS

At Brooklyn College: Associate Professors Samuel Borofsky and C. B. Boyer have been promoted to professorships; Mrs. Margaret Y. Woodbridge has been promoted to an assistant professorship; Professor Edward Fleisher has retired with the title of Professor Emeritus.

Brown University announces the following: Professor R. E. Gilman's leave of absence has been extended through 1953–54 in order that he may continue work in Washington; Mrs. Mildred Carlen Brunschwig has resigned from her position as Registrar of the Graduate School.

Cornell University reports the following: Professor J. B. Rosser has received a Fulbright Fellowship for the year 1953–54 and is studying at the University of Paris; Assistant Professor G. A. Hunt has been promoted to an associate professorship; Mr. Walter Feit, previously a teaching assistant at the University of Michigan, Dr. I. S. Gál, formerly at the Institute for Advanced Study, Assistant Professor J. K. Goldhaber of the University of Connecticut, Dr. C. S. Herz, formerly National Science Foundation Fellow at Princeton University, and Dr. Steven Orey, previously a teaching assistant at the University, have been appointed to instructorships; Mr. R. C. Lesser of the Office of Statistical Services of Massachusetts Institute of Technology has been appointed Director of the Cornell Computing Center; Professor Pierre Samuel has returned to his position at Clermont-Ferrand, France.

At Georgia Institute of Technology: Assistant Professor J. R. Garrett has been promoted to an associate professorship; Dr. B. M. Drucker, Dr. J. A. Nohel, and Dr. Henry Sharp, Jr., have been appointed to assistant professorships; Assistant Professor W. B. Evans and Instructor W. R. Carnes have returned to resume their teaching duties after two years with the Air Force.

Kansas State College announces: Dr. J. M. Marr, previously a teaching assistant at the University of Tennessee, and Assistant Professor W. L. Stamey of the University of Georgia, Atlanta Division, have been appointed to assistant professorships; Mr. Albert Derin, formerly a graduate student at the University of Chicago and Mr. J. V. Guida, who has been a graduate student at the University of Missouri, have been appointed to instructorships; Assistant Professor Albert Furman is on sabbatical leave for the year 1953–54 and is studying at the University of Kansas; Professor P. M. Young has been appointed acting Dean of Students for the current academic year.

Mills College reports the following: Assistant Professor Grayson Schmidt, formerly head of the Department of Mathematics and Physics has resigned; Dr. Andrewa R. Noble has been appointed Assistant Professor of Mathematics and Physics and Head of the Department.

Oberlin College announces the following: Chairman E. P. Vance is on sabbatical leave for the year 1953–54; Professor R. R. Stoll will serve as Acting Chairman of the Department of Mathematics during the year; in addition to appointments announced previously, Dr. J. D. Baum of Yale University and Dr. A. D. Martin of Washington University have been appointed to instructorships; during the past summer Professor Stoll assisted Professor Artin at the Summer Conference on Collegiate Mathematics at Boulder, Colorado.

University of Buffalo announces that Mr. D. O. McKay, formerly a teaching fellow at the University, and Mr. P. J. Schillo, who has been a graduate student at the University, have been appointed to instructorships.

At the University of Michigan: Assistant Professor R. C. Lyndon of Princeton University has been appointed to an assistant professorship; Instructor Edwin Weiss of Massachusetts Institute of Technology has been appointed to an instructorship; Associate Professor E. D. Rainville has been promoted to a

professorship; Assistant Professors P. S. Jones and George Piranian have been promoted to associate professorships; Instructor Frank Harary has been promoted to an assistant professorship; Assistant Professor N. H. Anning has retired with the title of Assistant Professor Emeritus; Associate Professor Wilfred Kaplan is on sabbatical leave for the year 1953–54; Associate Professor Hans Samelson is on leave for the year 1953–54 and is at the Institute for Advanced Study; Professor A. H. Copeland will be on sabbatical leave for the second semester of 1953–54.

Mr. Silvio Aurora, previously a graduate student at Columbia University, has a position as a tutor at Queens College.

Dr. Aaron Bakst has been appointed to an assistant professorship in the School of Commerce, New York University.

Acting Assistant Professor B. J. Ball of the University of Virginia has been appointed to an assistant professorship.

Dr. W. E. Barnes is now Head of the Ballistics and Statistical Theory Branch, Computation and Ballistics Department, United States Naval Proving Ground, Dahlgren, Virginia.

Dr. Max Beberman of the University of Illinois has been promoted to the position of Assistant Professor of Education.

Mr. V. W. Beck, formerly an electrical engineer with American Television and Radio, St. Paul, Minnesota, is now a research engineer with Minneapolis Honeywell Company.

Mr. J. S. Becker is now an engineering assistant in the Aviation Division, Studebaker Corporation, South Bend, Indiana.

Mr. P. R. Beesack has been appointed to a research assistantship at Washington University.

Mr. Jonas Beraru, previously a design engineer with Ford Instrument Company, Long Island City, New York, has accepted a position as a mathematician with the Reeves Instrument Corporation, New York City.

Dr. Gerald Berman of Illinois Institute of Technology has been promoted to an assistant professorship.

Mr. H. A. Bernhard, recently a graduate student at Columbia University, is now employed as a research engineer by the Boeing Airplane Company, Melbourne, Florida.

Miss Ida M. Bernhard, who has been Supervisor of Mathematics at Southwest Texas State Teachers College, is now Consultant in Secondary Education, Texas Education Agency, Austin, Texas.

Mr. H. H. Berry has accepted a position as a numerical analyst with the General Electric Company, Cincinnati, Ohio.

Associate Professor M. T. Bird of San Jose State College has been promoted to a professorship.

Mr. S. R. Bodner of Polytechnic Institute of Brooklyn has been appointed to a research assistantship.

Mr. J. R. Boyd, who has been teaching at San Marcos High School, Texas, has accepted a position at Chance Vought Aircraft, Dallas, Texas.

Dr. J. W. Brace, recently a graduate student at Cornell University, has been appointed to an instructorship at the University of Maryland.

Dr. W. E. Briggs of the University of Colorado has been appointed Research Assistant.

Dr. Paul Brock is now Head of Technical Services, Computer Division, Consolidated Engineering Corporation, Pasadena, California.

Professor D. M. Brown has been promoted to the position of Head of the Department of Data Reduction and Computation, Willow Run Research Center, University of Michigan.

Associate Professor H. K. Brown of Northeastern University has been promoted to the position of Professor of Mechanical Engineering and Director of Engineering Graduate Study.

Mr. A. L. Buchman, formerly of Hutchinson Central High School, Buffalo, New York, is teaching now at Technical High School, Buffalo.

Assistant Professor Emily E. Calkins of the College of William and Mary has been promoted to an associate professorship.

Miss Dorothy I. Carpenter of Denison University has been appointed to an associate professorship at Ashland College.

Mr. R. J. Cary, previously a student at Harpur College, has accepted a position with the Ansco Company, Binghamton, New York.

Mr. P. L. Chessin, formerly of Cooper Union School of Engineering, has a position as Associate Engineer in the Air Arm Division, Westinghouse Electric Corporation, Baltimore, Maryland.

Associate Professor D. E. Christie has been awarded a Faculty Fellowship from the Fund for the Advancement of Education; he is on leave of absence from Bowdoin College for the year 1953–54 and is Visiting Fellow in the Department of Mathematics, Princeton University.

Associate Professor B. G. Clark of Vanderbilt University has been promoted to a professorship.

Mr. M. J. Cleveland has a position as a mathematician in the Underwater Explosions Research Division, Norfolk Naval Shipyard, Portsmouth, Virginia.

Assistant Professor D. E. Coffey of Lawrence Institute of Technology has been promoted to an associate professorship.

Associate Professor L. W. Cohen is on leave of absence from Queens College and has been appointed Program Director for Mathematical Sciences for the National Science Foundation.

Assistant Professor C. H. Cook of Western State College, Austin, Texas, has accepted a position as Aerophysics Engineer with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. K. L. Cooke of State College of Washington has been promoted to an assistant professorship.

Professor N. A. Court of the University of Oklahoma has retired with the

title of Professor Emeritus.

Mr. D. E. Deal of Ball State Teachers College, Indiana, has been promoted to an assistant professorship.

Dr. W. E. Deskins of the University of Wisconsin has been appointed to an instructorship at Ohio State University.

Mr. David DeVol of the University of Colorado has accepted a position with the firm of George B. Buck, Consulting Actuary, New York City.

Mr. A. R. DiDonato, formerly with the du Pont Experimental Station, Wilmington, Delaware, is employed as a mathematical physicist at the Melpar Corporation, Alexandria, Virginia.

Dr. H. P. Edmundson, previously a graduate student at the University of California at Los Angeles, has accepted a position with the Department of Defense, Washington, D. C.

Dr. J. E. Flanagan of the University of Illinois has been appointed to an assistant professorship at the Carnegie Institute of Technology.

Mr. A. J. Flynn, formerly an assistant at Illinois State Normal University, has a position as an analyst in the Engineering Division, Eureka Williams Corporation, Bloomington, Illinois.

Dr. J. R. Foote, previously at the Wright-Patterson Air Force Base, Dayton, Ohio, has been appointed to an assistant professorship at the University of Oklahoma.

Mr. D. E. Freeland has been appointed Research Assistant in the Statistical Laboratory at Purdue University.

Mr. W. H. From, who has been with Aircraft Armaments, Inc., Baltimore, Maryland, has accepted a position at the American Machine and Foundry Company, Boston, Massachusetts.

Mr. G. A. Galloway, formerly head of the Department of Mathematics of Flat River Junior College, Missouri, is now Superintendent of Schools, Kane Unit Schools, Fairfield, Illinois.

Mr. R. H. Gillespie has accepted a position as Senior Research Assistant at the Electro-Metallurgical Company, Niagara Falls, New York.

Mr. W. H. Glenn, Jr., who has been Assistant Curriculum Coordinator for the Pasadena City Schools, has been appointed Chairman of the Department of Mathematics of Pasadena City College.

Dr. Herbert Goertzel, previously of the Oak Ridge National Laboratory, is now with the AEC Computing Facility, New York University.

Dr. Lillian Gough of the University of Buffalo has been appointed to an instructorship at Oswego State Teachers College, New York.

Mr. F. D. Grogan, who has been Resident Inspector for the Dallas Chemical Procurement District, Texas, is now Resident Inspector, Army Chemical Corps, Rocky Mountain Arsenal, Denver, Colorado.

Mr. H. M. Gurk, formerly a graduate student at the University of Pennsylvania, has been appointed to an instructorship in the Moore School of Electrical Engineering of the University.

Mr. F. S. Hawthorne of Hofstra College has been promoted to an assistant professorship.

Dr. C. M. Hebbert, who has retired from his position at the Bell Telephone Laboratories, has been appointed to a professorship at the Polytechnic Institute of Brooklyn.

Dr. Melvin Henriksen of Purdue University has been promoted to an assistant professorship.

Mr. D. M. Hester, who has been teaching at Liberty High School, Texas, has been appointed Head of the Department of Mathematics of Highland Junior College, Kansas.

Assistant Professor W. N. Huff of the University of Oklahoma has been promoted to an associate professorship.

Mr. O. C. Juelich, previously an assistant at Ohio State University, is engaged as an aerodynamicist with North American Aviation, Columbus, Ohio.

Mr. R. A. Kennedy, formerly a student at Illinois Institute of Technology, has accepted a position as Quality Control Supervisor with Johnson and Johnson, Chicago, Illinois.

Dr. R. J. Koch of Tulane University has been appointed to an assistant professorship at Louisiana State University.

Mr. Sidney Kravitz, previously at Picatinny Arsenal, Dover, New Jersey, has a position with Eastern Engineering Company, Hackensack, New Jersey.

Dr. R. J. Lambert of the National Security Agency, Washington, D. C., has accepted a position as Assistant Professor of Mathematics at Iowa State College; he also retains a position as Consultant with the Agency.

Mr. W. D. Lambert has a position as a consultant in the Mapping and Charting Research Laboratory, Columbus, Ohio.

Mr. David Loev, formerly a student at the University of Pennsylvania, has accepted a position as an engineer with the Burroughs Corporation, Philadelphia, Pennsylvania.

Associate Professor Lee Lorch of Fisk University has been promoted to a professorship.

Mr. R. A. Magda, previously a student at the University of Detroit, is teaching at Assumption High School, Windsor, Ontario, Canada.

Mr. J. N. Mangnall, formerly with Bell Aircraft Corporation, Niagara Falls, New York, is engaged as Associate Mathematician at Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. J. A. Mansour of the University of Detroit has been promoted to an instructorship.

Mr. W. A. Mary is teaching at Webster Junior High School, Collinsville, Illinois.

Brother Thomas Matthews is now President of Christian Brothers College, Memphis, Tennessee.

Mr. H. T. McAdams of the Aluminum Research Laboratories, East St. Louis, Illinois, has accepted a position as a research physicist with the Cornell

Aeronautical Laboratory, Buffalo, New York.

Mr. P. J. McCarthy has been appointed to a research assistantship at the University of Notre Dame.

Associate Professor A. W. McGaughey of Bradley University has been promoted to a professorship.

Mr. R. T. McLean of the College of Steubenville has been promoted to an assistant professorship.

Mr. F. A. McMahon, previously with Sperry Gyroscope Company, Great Neck, New York, is now a training supervisor at General Precision Laboratory, Inc., Pleasantville, New York.

Research Associate Paul Meier of Johns Hopkins University has been promoted to the position of Assistant Professor of Biostatistics.

Mr. H. E. Menke of Heidelberg College has accepted a position as a mathematician with the National Machinery Company, Tiffin, Ohio.

Mr. R. B. Merkel, who has been teaching in the Sacramento Unified School District, has been appointed to an assistant instructorship at Sacramento State College.

Mr. A. C. Moeller has been promoted to an assistant professorship in the Department of Electrical Engineering, Marquette University.

Assistant Instructor R. E. Montgomery of Rutgers University has been appointed to an instructorship at Trinity College, Connecticut.

Mr. A. J. Mortola of St. Peter's College has been appointed to an assistant professorship at Manhattan College.

Mr. T. D. Nagle, formerly a student at the University of Bridgeport, is employed now by Pratt and Whitney Aircraft Corporation, East Hartford, Connecticut.

Mr. R. J. Oravec, who was an actuarial student with Equitable Life Assurance Society of United States, New York City, is engaged now as an operations research scientist with Republic Aviation, Farmingdale, New York.

Dr. L. L. Pennisi of the University of Illinois, Navy Pier, Chicago, has been promoted to an assistant professorship.

Associate Professor H. H. Pixley of Wayne University has been appointed Associate Dean of Administration.

Mr. D. W. Pounder, previously a senior aerodynamicist at A. V. Roe Canada, Ltd., Toronto, has a position as Aerodynamic Engineer with deHavilland Propellers, Ltd., Hatfield, Hertfordshire, England.

Associate Professor L. L. Rauch of the University of Michigan has been promoted to the position of Professor of Instrumentation.

Dr. D. B. Ray, previously a graduate student at Cornell University, has been awarded a Jewett Fellowship and will study at Princeton University during 1953–54.

Mr. J. D. Rice, assistant at Rice Institute, has been appointed to an assistant professorship at Lamar State College of Technology.

Mr. M. B. Ritterman of Long Island University has a position as Senior

Engineer with Sylvania Electric Products, Inc., Bayside, New York.

Associate Professor Herbert Robbins of the University of North Carolina has been appointed Professor of Mathematical Statistics at Columbia University.

Dr. Louis Robinson, formerly a graduate student at Syracuse University, has accepted a position as a mathematician in the Applied Science Division, International Business Machines Corporation, Boston, Massachusetts.

Dr. Lawrence Rosenfeld has accepted a position with Raytheon Manufacturing Company, Waltham, Massachusetts.

Mr. D. T. Ross is engaged as a mathematician with the Servomechanisms Laboratory, Massachusetts Institute of Technology.

Mr. A. R. Schmidt, previously a teaching assistant at Purdue University, has been appointed to an assistant professorship at Rose Polytechnic Institute.

Professor D. M. Seward of Ouachita Baptist College has been appointed Dean of the Faculty.

Dr. D. H. Shaffer, formerly assistant research mathematician at Carnegie Institute of Technology, has been appointed to an instructorship at the Institute.

Sister Mary Charles of Immaculate Heart College has been promoted to an associate professorship.

Sister Mary Raphael of Immaculata College has been promoted to an assistant professorship.

Mr. A. T. Skinner of Champlain College has been appointed Dean of Auburn Community College, New York.

Associate Professor Ernst Snapper of the University of Southern California has been promoted to a professorship.

Mr. J. L. Solomon, formerly with Solomon and Sons, New York City, is now President of Universal Electronic Laboratories, Inc., New York City.

Dr. P. C. Squires has accepted a position in the Visual Engineering Laboratory, Medical Research, United States Naval Submarine Base, New London, Connecticut.

Mr. F. W. Stallard is on leave from East Tennessee State College and has a position as a graduate fellow at Oak Ridge Institute of Nuclear Studies.

Dr. R. D. Stalley, previously a graduate assistant at the University of Oregon, has been appointed to an instructorship at Iowa State College.

Instructor E. T. Stapleford of Kent State University has been promoted to an assistant professorship.

Dr. J. K. Sterrett, formerly at Aberdeen Proving Ground, Maryland, is now at the Naval Research Laboratory, Washington, D. C.

Mr. R. F. Steward of Virginia Military Institute has been appointed to a part-time instructorship at Drexel Institute of Technology.

Associate Professor A. T. Street of Roosevelt College has been promoted to a professorship.

Dr. W. J. Thomsen of State Teachers College, Whitewater, Wisconsin, has a

position at the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Miss Evelyn L. Trennt of Springfield Junior College, Illinois, has been appointed to an instructorship at Milwaukee Downer Seminary.

Mr. Carl Tross, previously at the Wright Air Development Center, Dayton, Ohio, is now with Lockheed Aircraft Corporation, Burbank, California.

Mr. S. V. S. Walker, formerly a student at The Citadel, has a position as Teacher and Assistant Principal at North Charleston High School, South Carolina.

Professor Emeritus F. B. Wiley of Denison University has resumed his teaching duties at the University.

Mr. L. R. Chase, a teacher at Rogers High School, Newport, Rhode Island, died on December 6, 1952. He had been a member of the Association for twenty-seven years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FOURTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The fourteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 6, 1954. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to undergraduate students in universities and colleges of the United States and Canada who have not received a college degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, Box 30, Kent State University, Kent, Ohio, by a postcard request. All applications must be filed with the Director not later than February 10th, 1954. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to the contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the thirteen previous competitions and examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August–September 1947, December 1948, August–September 1949, 1950, 1951, October 1952 and 1953.

CORRECTION

In the abstract *Orthic and oblique hyperbolas*, by V. B. Temple, this MONTHLY, vol. 60, 1953, p. 364, the following correction should be made; insert "Draw any line OA intersecting L_1 in T and L_2 in S.", between the first and second sentences of the first paragraph.

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: E. F. Allen, R. C. Archibald, J. P. Ballantine, P. T. Bateman, Garrett Birkhoff, S. S. Cairns, G. F. Cramer, A. B. Coble, R. P. Dilworth, C. L. Dolph, L. R. Ford, J. S. Frame, R. E. Greenwood, B. Jacobsohn, P. S. Jones, M. Kac, J. L. Kelley, S. C. Kleene, L. A. Kokoris, H. W. Kuhn, D. H. Lehmer, K. O. May, C. V. Newsom, C. O. Oakley, O. Ore, R. Peterson, E. Pitcher, E. L. Post, R. M. Redheffer, S. M. Ulam, R. M. Winger, R. C. Yates, H. S. Zuckerman.

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CALENDAR OF FUTURE MEETINGS

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 5, 1953.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, April, 1954.

MINNESOTA, Hamline University, St. Paul, May 8, 1954.

MISSOURI, University of Missouri, Columbia, Spring, 1954.

NEBRASKA

NORTHERN CALIFORNIA

OHIO, April, 1954.

OKLAHOMA

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April, 1954.

SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.

SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.

SOUTHWESTERN, Arizona State College, Tempe, April 16–17, 1954.

TEXAS, Texas Technological College, Lubbock, April, 1954.

UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

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Edited by HOWARD EVES, State University of New York, and E. P. STARKE, Rutgers University

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